Wisdom in Tian Ji's Horse Racing Strategy

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Abstract The philosophy behind Tian Ji's horse racing strategy, an interesting legendary Chinese story, has been analyzed mathematically. In this paper, the formulation of the generalized Tian Ji's horse racing strategy with different number of horses has been developed. Various winning, drawing or losing combinations and probabilities have been addressed. This may provide food for thought on business competitiveness or military struggle, in particular, the weaker overcoming the stronger.

Keywords: legendary Chinese story, decision making, Eulerian number.

1. Introduction

In the modern world today, the survival of the fittest holds for general people, but the survival of the cunning is possible for intellectuals. Nowadays, with the development of society, every company faces more intense competition, especially for weaker and smaller enterprises. In actuality, stronger and larger companies possess much more advantages and better talents than weaker and smaller ones. The situation applies to developing nations all over the world too. Every nation, every country, strong or weak, vies for a foothold in every arena of economy, sports, arts and so on. All aims to be the strongest and the best, but in actuality, the strong possesses much more advantages and better talents than the growing multitude of developing nations.

Despite the fact, the strongest of warriors has his Achilles' heel. In such a competitive world whereby almost everything can be rivaled in, it is necessary to plan the order of engagement or competition to overcome the stronger or to claim a greater victory. In economics, companies and government agencies allocate resources according to the needs and requirements of the situation, often giving the minimum budget for the success of a project to maximize profit. In another way, it is a way of conserving resources to cope with greater threats. On a personal level, when playing a group game, leader must choose his players in sequence. The sequence of the group game should be calculated properly, especially to encounter a stronger competitor. This procedure is very similar to Tian Ji's horse racing strategy, an interesting legendary Chinese story. In this paper, the generalized Tian Ji's horse racing strategy is analyzed mathematically with the aim of finding general clues which have the potential to be applied in the modern world today.

2. Tian Ji's Horse Racing Strategy

In ancient China, there was an era known as "Warring States Period" (403 BC – 221 BC) during which China was not a unified empire but divided by independent

Seven Warring States with conflicting interests, one of which was Qi State located in eastern China. From 356 BC to 320 BC, the ruler of Qi State was Tian Yin-Qi (378 BC – 320 BC), King Wei of Qi. The story of Tian Ji's horse racing strategy, which is well-known and popular in China today, was originally recorded (Si-Ma, 91 BC) in the biography of Sun Bin (? – 316 BC), as a military strategist in Qi State ruled by King Wei of Qi:

General Tian Ji, a high-ranked army commander in Qi State, frequently bet heavily on horse races with King Wei of Qi. Observing that their horses, divided into three different speed classes, were well-matched, Sun Bin then advised Tian Ji, "Go ahead and stake heavily! I shall see that you win." Taking Sun Bin at his word, Tian Ji bet a thousand gold pieces with the King. Just as the race was to start, Sun Bin counseled Tian Ji, "Pit your slow horse against the King's fast horse, your fast horse against the King's medium horse, and your medium horse against the King's slow horse." When all three horse races were finished, although Tian Ji lost the first race, his horses prevailed in the next two, in the end getting a thousand gold pieces from the King.

Amazingly, the victorious strategy (as did Tian Ji after following Sun Bin's advice) was remarkable to be solved 2300 years long before operations research and game theory were invented (Shu *et al.*, 2011). This was only one way that Tian Ji could claim a victory over the King, as illustrated in Fig. 1. All other ways would present Tian Ji with loss. Naturally, the Sun Bin's victorious advice, named Tian Ji's horse racing strategy, may be extended to the scenario that Tian Ji and the King would race horses with arbitrary N different speed classes. In order to facilitate the analysis of the generalized Tian Ji's N-horse racing strategy, the N horses owned by two players: Tian Ji (T) and the King (K) are denoted respectively by T_n and K_n, where the subscript $n = 1, 2, \cdots, N$ is defined as player T's or K's horse in the nth speed class. In this scenario of N-horse racing, T's horse in a faster class is able to beat K's horse in a slower class, but T's horse are unable to beat K's horse in the same or faster class. Without losing generality, the relative racing capabilities of horses are $T_{n+1} \prec K_{n+1} \prec T_n \prec K_n$ for any $n = 1, 2, \cdots, N-1$, where the symbol "$\prec$" means "unable to beat" and the larger subscript corresponds to the slower class.

Figure1: Tian Ji's horse racing strategy.

T and K would choose the same class of N-horse racing, that is, the pairwise racing is $\begin{pmatrix} T_1 & T_2 & \cdots & T_N \\ K_1 & K_2 & \cdots & K_N \end{pmatrix}$. Naturally, in view of T's horse slower than K's one in the same class ($T_n \prec K_n,\ n = 1, 2, \cdots, N$), T's horses would lose all races. The essence of Tian Ji's horse racing strategy is that the originally-classified racing appearance of T's horses should be shifted one place in order to achieve T's best result. The best strategy for the generalized Tian Ji's N-horse racing, suggested by Sun Bin, should be the pairwise racing $\begin{pmatrix} T_N & T_1 & \cdots & T_{N-1} \\ K_1 & K_2 & \cdots & K_N \end{pmatrix}$. T would claim a victory of the N-horse racing with one loss and $N-1$ wins.

If K always chooses the racing appearance $(K_1, K_2, \cdots, K_N)$ against the T's best response $(T_N, T_1, \cdots, T_{N-1})$, T would gain every bet. Naturally, K would soon realize that the racing appearance $(K_1, K_2, \cdots, K_N)$ is resulting in recurrent losses. K would become an active player and consider alternative racing appearance to turn the racing around. A competitive situation is encountered for each player competing with a total of $N!$ combinatorial pairwise racing $\begin{pmatrix} T_{\sigma(1)} & T_{\sigma(2)} & \cdots & T_{\sigma(N)} \\ K_1 & K_2 & \cdots & K_N \end{pmatrix}$ (where σ is the permutation) available to T and K. The above explanation indicates that T would lose all races for the unit permutation $\sigma(n) = n$ (as the T's worst strategy) and T would claim a victory with one loss and $N-1$ wins for the shift permutation $\sigma(n) = \begin{cases} N & n = 1 \\ n-1 & 1 < n \le N \end{cases}$ (as the T's best strategy). Then the natural question is what is T's winning probability for randomly-pairwise racing between T's and K's horses. The equivalent question is how many permutations are available to T as T's victorious strategies.

Theorem 1. *The number of T having exactly M wins in N-horse racing is Eulerian number,*

$$E(N, M) = \sum_{m=0}^{M} (-1)^m (M + 1 - m)^N \frac{(N+1)!}{m!\,(N+1-m)!},$$

the number of permutations on $\{1, 2, \cdots, N\}$ with exactly M excedances.

Proof. An excedance of the permutation σ on $\{1, 2, \cdots, N\}$ is defined as any index n such that, $\sigma(n) > n$ and Eulerian number $E(N, M)$ (Euler, 1755) is defined as the number of the permutation σ on $\{1, 2, \cdots, N\}$ with exactly M excedances (Rosen, 2000). It is obvious that the existence condition of T having one win is $K_n \prec T_{\sigma(n)}$, that is, $\sigma(n) > n$ for any index n. To determine the number of T's wins is equivalent to count the number of $\sigma(n) > n$, which is an excedance of the permutation σ in the parlance of combinatorics. So the number of T having exactly M wins in N-horse racing is Eulerian number $E(N, M)$. The theorem is proved. □

In view of the symmetry property of Eulerian number, that is, $E(N, M) = E(N, N - (M + 1))$, Theorem 1 can be expressed equivalently as

Theorem 2. *The number of T having exactly $M + 1$ no-wins in N-horse racing is Eulerian number.*

In the above two theorems, it is interesting to note that the generalized Tian Ji's horse racing strategy, as the extension of the famous legendary Chinese story, can be viewed as a practical demonstration of applying Eulerian number. There are only three outcomes for T, namely, winning combination

$$\begin{cases} \sum_{M=\frac{N+1}{2}}^{N-1} E(N,M) & \text{odd } N \\ \frac{N!}{2} - E(N, \frac{N}{2}) & \text{even } N \end{cases}$$

with probability

$$\begin{cases} \frac{1}{N!} \sum_{M=\frac{N+1}{2}}^{N-1} E(N,M) & \text{odd } N \\ \frac{1}{2} - \frac{1}{N!} E(N, \frac{N}{2}) & \text{even } N \end{cases},$$

drawing combination

$$\begin{cases} 0 & \text{odd } N \\ E(N, \frac{N}{2}) & \text{even } N \end{cases}$$

with probability

$$\begin{cases} 0 & \text{odd } N \\ \frac{1}{N!} E(N, \frac{N}{2}) & \text{even } N \end{cases},$$

or losing combination

$$\begin{cases} \sum_{M=0}^{\frac{N-1}{2}} E(N,M) & \text{odd } N \\ \frac{N!}{2} & \text{even } N \end{cases}$$

with probability

$$\begin{cases} \frac{1}{N!} \sum_{M=0}^{\frac{N-1}{2}} E(N,M) & \text{odd } N \\ \frac{1}{2} & \text{even } N \end{cases},$$

which are shown in the table below.

Table1: Combination and probability with variable number of horses.

		Combination (with probability)		
N horses	Total $N!$	Winning	Drawing	Losing
1	1	0 (0%)	0 (0%)	1 (100%)
2	2	0 (0%)	1 (50%)	1 (50%)
3	6	1 (17%)	0 (0%)	5 (83%)
4	24	1 (4%)	11 (46%)	12 (50%)
5	120	27 (23%)	0 (0%)	93 (78%)
6	720	58 (8%)	302 (42%)	360 (50%)
7	5040	1312 (26%)	0 (0%)	3728 (74%)
8	40320	4541 (11%)	15619 (39%)	20160 (50%)
9	362880	103345 (29%)	0 (0%)	259535 (72%)
10	3628800	504046 (14%)	1310354 (36%)	1814400 (50%)

Probabilities for odd- or even- numbered horses are plotted respectively in Figs. 2 and 3. From the results illustrated above, the probabilities follow Eulerian distribution and there are two detectable characteristics. First, the case of odd-numbered horse racing has no drawing, which drawing happens only in the case of even-numbered horse racing. Second, the losing probabilities of any even-numbered horse racing are always at the constant 50% regardless of horse number involved.

Figure2: Trend of probability for odd-numbered horses.

In the odd-numbered horse racing, the winning and losing probabilities converge to the constant 50% as horse number increases due to no drawing; while in the even-numbered horse racing, the winning and drawing probabilities converge to a constant 25% as horse number increases due to the constant 50% losing. Overall, the winning probability of the odd-numbered horse racing is much higher than that of the adjacent even-numbered cases, for example, 23% of $N = 5$ is much higher than 4% of $N = 4$ and 8% of $N = 6$.

This shows that an odd-numbered horse racing gives better an opportunity of winning than an even-numbered case. This implies that the best combat units should be odd-numbered. Of course, if the combat efficiency would be getting better as N increases, the difficulty of controlling much large N combat units would be encountered. No drawing occurs in the odd-numbered horse racing, which means that a decisive outcome must be reached instead of a stalemate. More importantly, Figs. 2 and 3 suggest that the more horses involved, the larger N, result in the higher winning probability. Philosophically, it is typically the epitome of winning in numbers.

3. Conclusion

This paper discusses the formulation of determining the winning, drawing and losing probabilities of the generalized Tian Ji's horse racing strategy for any given racing horse numbers. Based on Eulerian number, the way of calculating the combination of having M wins in N-horse racing is so straightforward, thereby enabling us to find the probability of winning an entire game by having more wins than losses. It

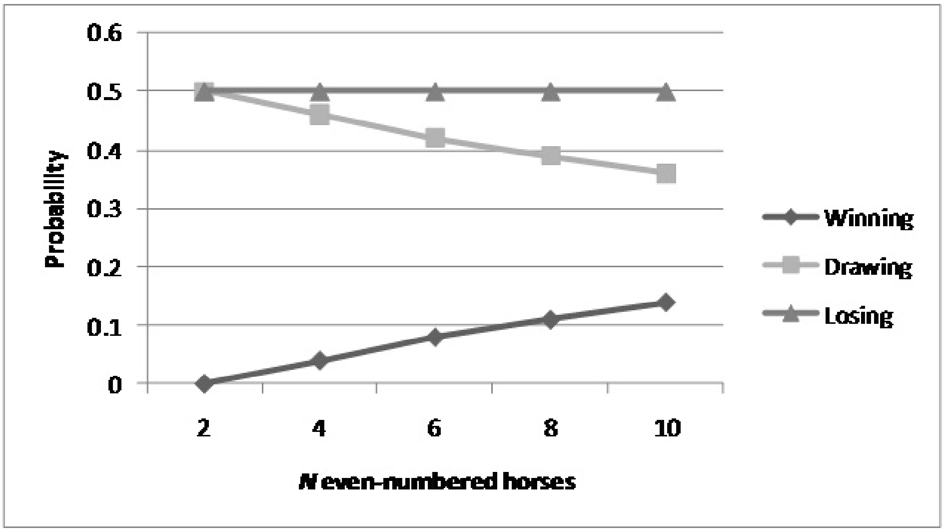

Figure3: Trend of probability for even-numbered horses.

is worth to mention to this end that that the larger N results in the higher winning probability. Philosophically, it is typically the epitome of winning in numbers.

Tian Ji's horse racing strategy is an interesting legendary Chinese story, which gives valuable insights to intellectuals that nothing is absolutely certain and thus nothing is impossible. Studying the theory of Tian Ji's horse racing strategy is very beneficial to society. In sports, this could be used in matching competitors in group games, such as, tennis, football, table tennis, badminton, *etc*. In economics, it could be used to allocate minimum resources to suitable tasks for optimizing profits. In engineering logistics, the results could be applied to arrange and transport goods with limited vehicles or manpower.

References

Si-Ma, Q. (91 BC). *Sun Zi Wu Qi Lie Zhuan (Biographies of Sun Zi and Wu Qi)*. In: Shi Ji (Records of History), **65(5)**, (in Chinese).

Shu, J.-J., Q.-W. Wang and K.-Y. Yong (2011). *DNA-based computing of strategic assignment problems*. Physical Review Letters, **106(18)**, 188702.

Euler, L. (1755). *Institutiones calculi differentialis cum eius vsu in analysi finitorum ac doctrina serierum*. Impensis Academiae Imperialis Scientiarum, Petropolitanae, (in Latin).

Rosen, K. H. (2000). *Handbook of Discrete and Combinatorial Mathematics*. CRC Press.