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Representations of the unitriangular group

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We find the generators of the field of invariants of codjoint representation for factors of the unitriangular Lie algebra with respect to an arbitrary regular ideal

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According to Kirillov's orbit method for nilpotent Lie groups, there exists one–to–one correspondence between coadjoint orbits and irreducible unitary representations in Hilbert spaces.

This make possible to solve many problems of representation theory and abstract harmonic analysis in terms of geometry of coadjoint orbits. However, for special nilpotent Lie groups, such as the unitriangular group, it turns out that classification of coadjoint orbits is still an open problem which is far from its solution [1], [2].

In his first paper on the orbit method [3], A. A. Kirillov classified coadjoint orbits of maximal dimension. Consider the family of orbits that do not annihilate the system of corner minors. One can prove that this family contains all orbits of maximal dimension (see [4]), and the algebra of invariants of the coadjoint action is a polynomial algebra generated by the system of corner minors.

In this talk we present generators of the field of invariants of the coadjoint representation of an arbitrary regular factor (see [5], [6], [7]). By definition, a regular factor is a factor Lie algebra of the unitriangular Lie algebra with respect to a regular ideal (i.e. an ideal stable under the adjoint action of the Cartan subgroup). If an ideal is zero, then our system of generators coincides with the system of corner minors of A. A. Kirillov.

Let $N = \mathrm{UT}(n,K)$ be the group of unitriangular group over the field K of zero characteristic. The elements of this group are the lower triangular matrices with units on the diagonal. The Lie algebra $\mathfrak{n} = \mathfrak{ut}(n,K)$ of this group consists of all lower triangular matrices with zeros on the diagonal. Define a natural representation of the group N on the conjugate space $\mathfrak{n}^*$ by the formula $\mathrm{Ad}_g^* f(x) = f(\mathrm{Ad}_g^{-1} x)$,where $f \in \mathfrak{n}^*$, $x \in \mathfrak{n}$ and $g \in N$. This representation is called the coadjoint representation. We identify the symmetric algebra $S(\mathfrak{n})$ with the algebra of regular functions $K[\mathfrak{n}^*]$

on the conjugate space $\mathfrak{n}^*$. Let us also identify $\mathfrak{n}^*$ with the subspace $\mathfrak{n}^+$ of upper triangular matrices with zeros on the diagonal. Pairing $\mathfrak{n}$ and $\mathfrak{n}^*$ is given by the Killing form $(a,b) = \mathrm{tr}\,(ab)$. Then the coadjoint representation is realized on $\mathfrak{n}^+$ by the formula $\mathrm{Ad}^*_g\, b = \Pi(\mathrm{Ad}_g\, b)$, where $b \in \mathfrak{n}^+$, $g \in N$, and Π is the natural projection of the space of $n \times n$ matrices onto $\mathfrak{n}^+$.

To simplify language we shall give the following definition: a root is an arbitrary pair (i,j), where i,j are positive integers from 1 to n and $i \neq j$. The permutation group S_n acts on the set of roots by the formula $w(i,j) = (w(i), w(j))$.

A root (i,j) is positive if $i > j$. Respectively, a root is negative if $i < j$. We denote the set positive roots by Δ^+.

We consider the ordering on Δ^+ such that

$$(n,1) \succ (n-1,1) \succ \ldots \succ (2,1) \succ (n,2) \succ \ldots \succ (n,n-1).$$

The (ij)-matrix units $\{y_{ij}\}$, $(i,j) \in \Delta^+$, form a basis (the standard basis) of the Lie algebra $\mathfrak{n}$. Let $\mathfrak{m}$ be a regular ideal in $\mathfrak{n}$. By definition, $\mathfrak{m}$ is spanned be some subsystem $\{y_{ij}\}$, $(i,j) \in M$, of standard basis. We refer factor algebra $\mathcal{L} = \mathfrak{n}/\mathfrak{m}$ as a regular factor.

Our first goal is to calculate the index of $\mathcal{L}$. Recall that the index of a Lie algebra is a minimal dimension of stabilizer of a linear form on it. For an algebraic Lie algebra the index coincides with the transcendental degree of the field of invariants of the coadjoint representation.

By means of the ideal $\mathfrak{m}$ we construct the diagram $D(\mathcal{L})$ that is a $n \times n$ matrix in which all places (i,j), $i \leqslant j$, are not filled and all other places (i.e. places of Δ^+) are filled by the symbols "$\otimes$", "$\bullet$", "$+$" and "$-$" according to the following rules. The places $(i,j) \in M$ are filled by the symbol "$\bullet$". We shall refer the procedure of placing of "$\bullet$" onto the places in M as the zero step in construction of the diagram.

We put the symbol "$\otimes$" on the greatest (in the sense of order $\succ$) place in $\Delta^+ \setminus M$. Suppose that we put the symbol "$\otimes$" on the place (k,t), $k > t$. Further, we put the symbol "$-$"on all places (k,a), $t < a < k$, and we put the symbol "$+$"on all places (b,t), $1 < b < k$. This procedure finishes the first step of construction of diagram. Further, we put the symbol "$\otimes$" on the greatest (in the sense of order $\succ$) empty place in Δ^+. As above, we put the symbols "$+$" and "$-$" on empty places, taking into account the following: we put the symbols "$+$" and "$-$" in pairs; if the both places (k,a) and (a,t), where $k > a > t$, are empty, we put "$-$" on the first place and "$+$" on the second place; if one of these places, (k,a) or (a,t), are already filled, then we do not fill the other place. After this procedure we finish the step which we call the second step.

Continuing the procedure further we have got the diagram. We denote this diagram by $D(\mathcal{L})$. The number of last step is equal to the number of the symbols "$\otimes$" in the diagram.

Example 1. Let $n = 7$, $\mathfrak{m} = Ky_{51} \oplus Ky_{61} \oplus Ky_{71} \oplus Ky_{62}$. The corresponding diagram is as follows

$$D(\mathcal{L}) = \begin{array}{|c|c|c|c|c|c|c|}
\hline
 & & & & & & \\
\hline
+ & & & & & & \\
\hline
+ & + & & & & & \\
\hline
\otimes & - & - & & & & \\
\hline
\bullet & + & + & \otimes & & & \\
\hline
\bullet & \otimes & - & + & - & & \\
\hline
\bullet & \bullet & \otimes & \otimes & - & - & \\
\hline
\end{array}$$

Theorem 1 *1) The maximal dimension of a coadjoint orbit in $\mathcal{L}^*$ equals to the number of symbols "$+$" and "$-$" in the diagram $D(\mathcal{L})$.*
2) The index of Lie algebra $\mathcal{L}$ coincides with the number of symbols "$\otimes$" in the diagram $D(\mathcal{L})$.

Consider a matrix $F(\mathcal{L})$, in which all elements on the places of M and on the places over the diagonal are zeroes, and all other places are filled by the corresponding elements of standard basis. In the case of Example 1 the matrix $F(\mathcal{L})$ has a form

$$F(\mathcal{L}) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
y_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\
y_{31} & y_{32} & 0 & 0 & 0 & 0 & 0 \\
y_{41} & y_{42} & y_{43} & 0 & 0 & 0 & 0 \\
0 & y_{52} & y_{53} & y_{54} & 0 & 0 & 0 \\
0 & y_{62} & y_{63} & y_{64} & y_{65} & 0 & 0 \\
0 & 0 & y_{73} & y_{74} & y_{75} & y_{76} & 0
\end{pmatrix}$$

We consider the characteristic matrix $F(\mathcal{L}) - \lambda E$. Notice that any minor of the characteristic matrix is a polynomial in λ with coefficients in $S(\mathfrak{n}) = K[\mathfrak{n}^*]$.

Definition 4 *A nonzero minor $M_I^J(\lambda)$ of the characteristic matrix $F(\mathcal{L}) - \lambda E$ is extremal if its degree decreases while its rows are moving down or its columns are moving to the left.*

Theorem 2 *The leading coefficient of an extremal minor is an invariant of the coadjoint representation of $\mathcal{L}$.*

Hypothesis. The algebra of invariants of the the coadjoint representation of Lie algebra $\mathcal{L}$ is generated by leading coefficients of extremal minors.

Our next goal is to assign to any symbol "$\otimes$" an extremal minor. First, we assign to any symbol "$\otimes$ that lies in the i-th column and in the j-th row, the positive root (i, j). We obtain the chain of roots $S = \{\xi_1 \succ \ldots \succ \xi_s\}$. For any $1 \leqslant i \leqslant s$

we construct a minor of the $M_i(\lambda)$ of the characteristic matrix with the system of rows $I(i)$ and columns $J(i)$. Consider a permutation $w_i = r_1 \ldots r_i$, where each r_j is a reflection with respect to ξ_j. Let $\xi_i = (k,t)$. Then $J(i) = \{1 \leqslant j \leqslant t: \ w_i(t) \leqslant w_i(j)\}$. If $w_i(t) > t$, then we we put $I(i) = wJ(i)$. If $w_i(t) \leqslant t$ (indeed this implies $w_i(t) < t$), then we put $I(i) = I_*(i) \cup [w_i(t), t]$ where $I_*(i) = \{t \leqslant j \leqslant n: \ w_i(j) < w_i(t)\}$.

Proposition 1 *Any minor $M_i(\lambda)$, $1 \leqslant i \leqslant s$, is extremal. Its leading coefficient P_i is an invariant of the coadjoint representation of $\mathcal{L}$.*

Theorem 3 *The field of invariants of the coadjoint representation of the regular factor $\mathcal{L}$ is a field of rational functions of $\{P_i: \ 1 \leqslant i \leqslant s\}$.*

In Example 1 the field of invariants is generated by five polynomials

$$P_1 = y_{41}\,, \quad P_2 = y_{62}\,, \quad P_3 = y_{73}\,, \quad P_4 = y_{74}y_{41} + y_{73}y_{31}\,, \quad P_5 = \begin{vmatrix} y_{52} & y_{53} & y_{54} \\ y_{62} & y_{63} & y_{64} \\ 0 & y_{73} & y_{74} \end{vmatrix}.$$

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