МАТЕМАТИЧЕСКАЯ ЛОГИКА, АЛГЕБРА И ТЕОРИЯ ЧИСЕЛ И ДИСКРЕТНАЯ МАТЕМАТИКА / MATHEMATICAL LOGIC, ALGEBRA AND NUMBER THEORY AND DISCRETE MATHEMATICS

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ON A BOUNDARY PROBLEM IN BOHR SPACE OF MULTIVARIATE ALMOST PERIODIC FUNCTIONS

Research article

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Abstract

There are many applications of the theory of integral equations of Fredholm type. In the literature, applications of Fredholm theory to the questions of boundary problems of ordinary differential or equations with partial derivatives are best known. In this work, we consider the analogs of some problems of the theory of differential equations in Bohr spaces, solutions of which could be found by application of the theory of limit Fredholm equations in Bohr spaces of almost periodic functions. Differential equations are not solvable in the space of almost periodic functions, in general. By this reason, we modify posing of the problems with the aim that the question could solvable in Bohr space. In the work, we state the equivalent variant of boundary problems in the space of almost periodic functions.

Keywords: integral equations, Fredholm theory, almost periodic functions, Bohr spaces, boundary problems.

О КРАЕВОЙ ЗАДАЧЕ В БОРОВСКОМ ПРОСТРАНСТВЕ МНОГОЗНАЧНЫХ ПОЧТИ ПЕРИОДИЧЕСКИХ ФУНКЦИЙ

Научная статья

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Аннотация

Существует множество применений теории интегральных уравнений вида Фредгольма. В литературе наиболее известны случаи применения теории Фредгольма к вопросам краевых задач обыкновенных дифференциальных уравнений или уравнений с частными производными. В данной работе мы рассмотрим аналоги некоторых задач теории дифференциальных уравнений в пространствах Бора, решения которых могут быть найдены путем применения теории предельных уравнений Фредгольма в пространствах Бора почти периодических функций. Дифференциальные уравнения в пространстве почти периодических функций, как правило, не разрешимы. По этой причине мы модифицируем постановку задач с таким расчетом, чтобы вопрос мог быть разрешим в пространстве Бора. В работе излагается эквивалентный вариант краевых задач в пространстве почти периодических функций.

Ключевые слова: интегральные уравнения, теория Фредгольма, почти периодические функции, пространства Бора, краевые задачи.

Introduction

There are many applications of the theory of integral equations of Fredholm type. In the literature, applications of Fredholm theory to the questions of boundary problems of ordinary differential or equations with partial derivatives are best known. In this work, we consider some problems of the theory of differential equations, solutions of which could be found by application of the theory of limit Fredholm equations in Bohr spaces of almost periodic functions [3], [4], [5], [6]. In this paper, we will consider the analogs of two boundary problems and solve them by the theory of limit Fredholm equations.

In general, differential equations are not solvable in the space of almost periodic functions. By this reason, we modify posing of the problems with the aim that the question could solvable in Bohr space. In the work, we state the equivalent variant of boundary problems in the space of almost periodic functions. The method of solution is based on the construction of the analog of Green function corresponding to the boundary problem and leading the problem to the solution of limit integral equations of Fredholm type.

It is best known that the integral equations of Fredholm are closely connected with differential equations of first order

$$y' = f(x, y).$$

Many investigations were devoted to such equations in different classes of functions. In the Favard theory, for example, the special cases of these equations were considered for the class of Bohr almost periodic functions which belong to the class of continuous bounded functions. In some cases there are not almost periodic solutions for the equation

$$y' + A(x)y = f(x)$$

(see [8]) in which the functions $f(x)$ and $A(x)$ are almost periodic.

There are examples ([9]) showing existence of unbounded almost periodic functions in Besicovitch sense which cannot be the solution to any differential equations of a view

$$F\left(y, y', \ldots, y^{(n)}\right) = 0$$

with continuous function F.

In this work, we consider some problem of the theory of differential equations, solutions of which could be found by application of the theory of limit Fredholm equations in Bohr spaces of almost periodic functions [3], [4], [5], [6]. In the paper, we consider the analog of a boundary problem and solve it using the theory of limit Fredholm equations. Since differential equations are not solvable in the space of almost periodic functions in general, we modify the posing of the problems with the aim that the question was solvable in Bohr space.

Research methods and principles

Let *K(x, y)* be a bivariate almost periodic function, λ is a complex number. In [4] the following analogs of Fredholm functions are introduced:

$$D(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{b_n \lambda^n}{n!}$$

with

$$b_n = (-1)^n \lim_{T\to\infty} \frac{1}{T^n} \int_0^T \cdots \int_0^T \begin{vmatrix} K(x_1,\xi_1) & K(x_1,\xi_2) & \cdots & K(x_1,\xi_n) \\ K(x_2,\xi_1) & K(x_2,\xi_2) & \cdots & K(x_2,\xi_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(x_n,\xi_1) & K(x_n,\xi_2) & \cdots & K(x_n,\xi_n) \end{vmatrix} d\xi_1 d\xi_2 \cdots d\xi_n.$$

We also denote:

$$D_k(x, y; \lambda) = \lambda D(\lambda) K(x; y) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{Q_n(x;y)\lambda^{n+1}}{n!}; x, y \in \mathrm{R},$$

in which we set:

$$Q_n(x,y) = -\lim_{T\to\infty} \frac{n}{T^n} \int_0^T \cdots \int_0^T P_n(x, \xi, \xi_1, \dots, \xi_{n-1}) K(\xi, y) d\xi d\xi_1 \cdots d\xi_{n-1},$$

where

$$P_n(x, \xi, \xi_1, \dots, \xi_{n-1}) = \begin{vmatrix} K(x,\xi) & K(x,\xi_1) & \cdots & K(x,\xi_{n-1}) \\ K(\xi_1,\xi) & K(\xi_1,\xi_1) & \cdots & K(\xi_1,\xi_{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ K(\xi_{n-1},\xi) & K(\xi_{n-1},\xi_1) & \cdots & K(\xi_{n-1},\xi_{n-1}) \end{vmatrix}.$$

the functions $D(\lambda)$ and $D(x,\xi;\lambda)$ are integral functions of the variable λ. Let us consider limit integral equation of a view:

$$\varphi(x) = f(x) + \lambda \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,\xi)\varphi(\xi) d\xi \qquad (1)$$

where the function $f(x)$ is an almost periodic function, and the kernel $K(x,\xi)$ is a symmetric bivariate almost periodic function.

In [4] it was established that every equation of this type is equivalent to some ordinary integral equation in unite cube. The dimension of the cube is defined by the number of linearly independent Fourier exponents, if the set of them is finite (infinite dimensional case reduces to the sequence of finite cases). On this base, we can prove the lemma below.

Lemma 1. Let λ be a real number such that $D(\lambda) \neq 0.$ Then the equation (1) has a unique solution which is defined by the formula

$$\varphi(x) = f(x) + \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\xi) \frac{D(x,\xi;\lambda)}{D(\lambda)} d\xi. \qquad (2)$$

Main results

Let us consider some boundary problem in three-dimensional spaces. In the work [3], [7] it was considered the equation

$$u_{xx} + u_{yy} + u_{zz} + \frac{2\alpha}{x} u_x = 0, 0 < 2\alpha < 1, x > 0. \qquad (3)$$

In that works it was constructed the theory of potentials. For applications to boundary problems, was got the integral equation of Fredholm type. Using this method, we, as above, will introduce the boundary problem in Bohr space of almost periodic functions and will solve it using the theory of limit integral equations.

Let us consider, in brief, the method of the work [3]. Let be a Lyapunov surface in the half space $x > 0$ bounded by simple connected open domain X on the plane x=0. Denote by $x = x(s,t), y = y(s,t), z = z(s,t)$ the parametric equation of t γ he surface, and $(s,t) \in \bar{\Phi}, \Phi = \{(s,t) \mid 0 \le s \le 1, 0 \le t \le 1\}$ Denote by γ the common boundary of domains X and. Suppose that:

1) The functions $x = x(s,t), y = y(s,t), z = z(s,t)$ have continuous partial derivatives which does not vanish simultaneously;

2) when the points of Γ tends to γ , then the surfaces intersect under right angle.

Consider now the boundary problem of Dirichlet. It is required to find in D the solution of the equation (3), being continuous in, and satisfying the boundary conditions:

$$u|_\Gamma = \varphi(s,t), (s,t) \in \bar{\Phi}, u(0,s,t) = \tau_1(y,z), (y,z) \in \bar{X}, \qquad (4)$$

where $\varphi(s,t)$ and $\tau_1(s,t)$ are given functions for which $\varphi(s,t)|_\gamma = \tau_1(y,z)|_\gamma$. On X we take $(y,z) = (s,t)$

The solution of this equation is searched as a potential, with unknown density: $w_2(x,y,z) = \iint_\Gamma \mu_2(\theta,\vartheta) B_v^\alpha [q_2(\xi,\eta,\zeta;x,y,z)]\, d\theta d\vartheta,$

where $B_v^a\left[q_2(\xi,\eta,\zeta;x,y,z)\right]$ $(v=1,2)$ is a fundamental solution of the equation (3). Using the fact that for the satisfaction of the boundary condition it is required the equality $w_2(x,y,z)=\varphi_2(s,t),(x,y,z)\in\Gamma,$ we arrive at the integral equation for unknown density

$$\mu_2(s,t)-2\iint_\Gamma \mu_2(\theta,\vartheta)K_2(s,t,\theta,\vartheta)d\theta d\vartheta=-2\varphi_2(s,t) \quad (5)$$

in which $K_2(s,t,\theta,\vartheta)=B_v^\alpha\left[q_2(\xi(\theta,\vartheta),\eta(\theta,\vartheta),\zeta(\theta,\vartheta);x(s,t)y(s,t),z(s,t))\right]$.

Using the method of the work [4], we can formulate the analog of Dirichlet boundary problem in Bohr spaces. We suffice with posing and scheme of solution of the problem. For this, we take some triple of real (irrational) numbers (ρ,θ,ϑ) independent over the field of rational numbers. We put on X

$(s,t)=(\{\delta t\},\{\lambda t\}),t\in\mathrm{R}$

To every continuous function $f(s,t)$ from the Lebesgue class $\mathrm{L}_2(0,1)$ we put in correspondence almost periodic function $f(\{\delta t\},\{\lambda t\})$. The analog of integral equation (5) will be a limit integral equation

$$\mu_2(\delta,\lambda t)-2\lim_{T\to\infty}\frac{1}{T}\int_0^T\mu_2(pt,\lambda t)K_2(\delta t,\lambda t,\delta p,\lambda p)dp=-2\varphi_2(\delta t,\lambda t). \quad (6)$$

Substituting found density into the equality

$$w_2(x,y,z)=\lim_{T\to\infty}\frac{1}{T}\int_0^T\mu_2(\delta,\lambda t)B_v^\alpha\left[q_2(\xi,\eta,\zeta;x,y,z)\right]dt,$$

we obtain the required solution, satisfying the boundary conditions (for which we must substitute $x=x(s,t),y=y(s,t),z=z(s,t),$ $(s,t)=(\{\delta\}\},\{\lambda t\}))$. We get the potential $w_2(x,y,z)$ which will be a solution of the equation $u_{xx}+u_{yy}+u_{zz}+\frac{2\alpha}{x}u_x=0$ with boundary conditions (5), written out in the limit form

$$u|_\Gamma=\lim_{m\to\infty}\lim_{n\to\infty}\varphi\left(t_m\delta,t_m\lambda\right); \quad (7)$$

$$u(0,y,z)=\lim_{m\to\infty}\lim_{n\to\infty}\tau_1\left(y\left(t_m\delta,t_m\lambda\right),z\left(t_m\delta,t_m\lambda\right)\right), \quad (8)$$

where (t_m) is some sequences of real numbers tending to $+\infty$.

Note that in the work [7] it was proven that 2 is not an eigenvalue of the equation (6). So, the unique solution of the equation (6) can be found by using of resolvent of the limit integral equation, applying the theorem 3.2 of the work [4].

Let us transform the given equation by such manner that to get a boundary problem in Bohr space of multivariate almost periodic functions. We have

$$u_t=\frac{1}{\rho}u_{\rho t}=\frac{1}{\rho}u_x.$$

Analogically,

$$u_{tt}=\frac{1}{\rho^2}u_{xx},u_{tt}=\frac{1}{\theta^2}u_{yy},u_{tt}=\frac{1}{\vartheta^2}u_{zz}.$$

So, we can rewrite given equation as follows

$$\left(\rho^2+\theta^2+\vartheta^2\right)u_{tt}+\frac{2\alpha\rho}{x}u_t=0. \quad (9)$$

Resuming all of said above, we have proven the main result of the paper.

Theorem. Let be a Lyapunov surface in the half space $x>0$ bounded by simple connected open domain X on the plane x=0 and the surface. Denote by $x=x(s,t),y=y(s,t),z=z(s,t)$ the parametric equation of the surface Γ and $(s,t)\in\Phi,\Phi=\{(s,t)\mid 0\le s\le 1,0\le t\le 1\}$.

Denote by γ the common boundary of domains X and, satisfying conditions 1)-2) above. Then there exists a function $u(x,y,z)$ such that for every triple (ρ,θ,ϑ) of real numbers, linearly independent over the field of rational numbers, the function $u(\{\rho t\},\{\theta\},\{\vartheta t\})$ is an almost periodic solution of the equation (9), satisfying the boundary conditions (7)-(8).

Conclusion

Note that the solutions of the equation (3) can be found as a limit value of obtained solution of the equation (9), due to everywhere denseness of values of almost periodic functions. This gives another point of a view to the boundary problems, because the equation (9) is simpler as the equation (3). This equation can be reduced to ordinary differential equations of first order, which can be taken by integration.

Simplification of the equation (3) can be performed for other equations with partial derivatives and boundary conditions, which are solvable by the construction of Green function.

Конфликт интересов

Не указан.

Рецензия

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Conflict of Interest

None declared.

Review

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