On limits of algebraic subgroups

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We shall consider complex algebraic varieties and groups. Both analytic and Zariski topology will be used. Unless stated otherwise, the analytic topology is meant.

For elements $g_1, \dots, g_p$ of an algebraic group G, we denote by $\langle g_1, \dots, g_p \rangle$ the Zariski closure of the subgroup generated by $g_1, \dots, g_p$. If $\langle g_1, \dots, g_p \rangle = G$, we say that G is *Zariski generated by* $g_1, \dots, g_p$. Any algebraic group is Zariski generated by finitely many elements. In particular, any connected reductive group is Zariski generated by two elements [Vi]. Closeness of algebraic subgroups can be evaluated by the closeness of their Zariski generating sets.

As usually, we denote the tangent Lie algebras of Lie groups $G, H, \dots$ by the corresponding gothic letters $\mathfrak{g}, \mathfrak{h}, \dots$

We set $G^p = \underbrace{G \times \cdots \times G}_{p}$.

Let H be an algebraic subgroup of an algebraic group G, and $g_1, g_2, \dots \in G$. Suppose that there exists a limit

$$\mathfrak{l} = \lim \mathrm{Ad}(g_n)\mathfrak{h}$$

in the relevant Grassmanian, and set

$$L = \lim g_n H g_n^{-1} = \{\lim g_n h_n g_n^{-1} : h_1, h_2, \dots \in H\}.$$

(Here $h_1, h_2, \dots$ are supposed to be chosen in such a way that $\lim g_n h_n g_n^{-1}$ should exist.) Obviously, L is a subgroup of G.

Theorem 1. *L is an algebraic subgroup with tangent algebra $\mathfrak{l}$.*

Theorem 2. *If H is reductive, then any reductive algebraic subgroup $S \subset L$ is conjugate to a subgroup of H.*

Page and Richardson [PR] proved the following stability property of semisimple subalgebras of Lie algebras:

Let $\mathfrak{s}$ be a semisimple subalgebra of a Lie algebra $\mathfrak{h}$, and $\mathfrak{h}'$ a sufficiently small deformation of $\mathfrak{h}$. Then there exists a subalgebra $\mathfrak{s}' \subset \mathfrak{h}'$ which is isomorphic to $\mathfrak{s}$ and close to $\mathfrak{s}$ (as a subspace).

A simple proof of this was given by Neretin [Ne, Lemma in Section 1.4].

Since close semisimple subalgebras of a Lie algebra are conjugate, the above stability property implies the following theorem:

(*) *Let $\mathfrak{s}$ be a semisimple subalgebra of a Lie algebra $\mathfrak{g}$. Then any subalgebra of $\mathfrak{g}$, containing a subspace of dimension $\dim \mathfrak{s}$ sufficiently close to $\mathfrak{s}$, contains a subalgebra conjugate to $\mathfrak{s}$ and close to $\mathfrak{s}$.*

Making use of this theorem, we prove the following version of it for algebraic groups.

Theorem 3. *Let S be a connected semisimple subgroup of an algebraic group G, and $\mathbf{s} = (s_1, \ldots, s_p)$ a Zariski generating set of S. For any neighbourhood U of e in G there exists a neighbourhood V of $\mathbf{s}$ in G^p such that any reductive subgroup $H \subset G$ satisfying the condition $H^p \cap V \neq \varnothing$, contains a subgroup gSg^{-1} with $g \in U$.*

It seems that the assumption on reductivity of H in Theorems 2 and 3 is superfluous, but I cannot avoid it.

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1. A property of the exponential mapping. Let $G \subset \mathrm{GL}_n(C)$ be an algebraic linear group.

Proposition 1. *There exists a positive number $c = c_G \leqslant \pi$ such that the exponential mapping*

$$\exp : \mathfrak{g} \to G$$

maps diffeomorphically the (open) set $U(\mathfrak{g}, c)$ of elements of $\mathfrak{g}$, whose eigenvalues λ satisfy the condition $|Im\lambda| < c$, onto the (open) set $U(G, c)$ of elements of G, whose eigenvalues λ satisfy the condition $|\arg \lambda| < c$.

Proof. The assertion is known for $\mathrm{GL}_n(C)$, with $c = \pi$ [MN]. It follows that for any positive $c \leqslant \pi$ the exponential mapping maps diffeomorphically $U(\mathfrak{g}, c)$ onto an open subset of $U(G, c)$. We are to prove that, for some c,

$$\xi \in U(\mathfrak{gl}_n(C), c) \ \& \ \exp \xi \in G \quad \text{implies} \quad \xi \in \mathfrak{g}. \tag{1}$$

Let $\xi = \xi_s + \xi_n$ be the additive Jordan decomposition of ξ. Then $\exp \xi = \exp \xi_s \cdot \exp \xi_n$ is the multiplicative Jordan decomposition of $\exp \xi$. If $\xi \in U(\mathfrak{gl}_n(C), c)$, then $\xi_s, \xi_n \in U(\mathfrak{gl}_n(C), c)$, and if $\exp \xi \in G$, then $\exp \xi_s, \exp \xi_n \in G$. So it suffices to prove (1) for semisimple and nilpotent elements.

If ξ is nilpotent and $\exp \xi \in G$, then $\exp t\xi \in G$ for any $t \in C$ and hence $\xi \in \mathfrak{g}$.

Let now ξ be semisimple. If the element $\exp \xi$ belongs to the connected component of G containing the unit, then it belongs to a maximal torus T of G. In a basis consisting of weight vectors, T is defined by equations of the form

$$\prod_j \lambda_j^{n_{ij}} = 1 \quad (i = 1, \ldots, m; \quad n_{ij} \in Z)$$

in the diagonal entries (eigenvalues) λ_j 's. The tangent algebra $\mathfrak{t}$ of T is defined by the equations

$$\sum_j n_{ij} \lambda_j = 0 \quad (i = 1, \ldots, m). \tag{2}$$

Obviously, ξ is diagonal in the same basis. Let $\lambda_1, \ldots, \lambda_n$ be its eigenvalues. Then

$$\prod_j (e^{\lambda_j})^{n_{ij}} = e^{\sum_j n_{ij}\lambda_j} = 1 \quad (i = 1, \ldots, m),$$

whence

$$\sum_j n_{ij} \lambda_j \equiv 0 \quad (mod 2\pi\sqrt{-1}), \quad (i = 1, \ldots, m). \tag{3}$$

If $\xi \in U(\mathfrak{gl}_n(C), c)$ with

$$c \leqslant \frac{2\pi}{\max_i \sum_j |n_{ij}|}, \tag{4}$$

then (3) implies (2), i.e., $\xi \in \mathfrak{g}$.

If the element $\exp\xi$ belongs to another connected component of G, say, G_1, then it belongs to a maximal toric subvariety S of G_1 (see [Vi]), which is still diagonal in some basis and is defined by equations of the form

$$\prod_j \lambda_j^{p_{ij}} = \mu_i \quad (i = 1, \ldots, m; \quad p_{ij} \in Z),$$

where μ_i 's are some roots of 1, not all of them being equal to 1. Obviously, ξ is diagonal in the same basis. Let $\lambda_1, \ldots, \lambda_n$ be its eigenvalues. Then

$$\prod_j (e^{\lambda_j})^{p_{ij}} = e^{\sum_j p_{ij}\lambda_j} = \mu_i \quad (i = 1, \ldots, m). \tag{5}$$

Let μ_k be a primitive q-th root of 1, where $q > 1$. If $\xi \in U(\mathfrak{gl}_n(C), c)$ with

$$c < \frac{2\pi}{q \sum_j |p_{kj}|}, \tag{6}$$

then (5) cannot be satisfied, which is a contradiction.

Thus, if c satisfies the inequalities (4) or (6) for all connected components of G, the implication (1) holds. □

2. A criterion for algebraicity of a complex Lie group. Let G be a complex Lie group, and G_0 its connected component containing the unit.

It is known that, for an algebraic group G,

(A) if $g = g_s g_u$ is the (multiplicative) Jordan decomposition of an element $g \in G$, then $g_s \in G, g_u \in G_0$.

Let us call a semisimple linear operator *compact*, if its eigenvalues are of modulus 1, and *positive*, if they are positive. Any semisimple linear operator g can be uniquely represented in the form

$$g = g_c g_p, \tag{7}$$

where g_c (resp. g_p) is a compact (resp. positive) semisimple linear operator and $g_c g_p = g_p g_c$. Let us call (7) the *polar decomposition* of g.

It is easy to see that, for an algebraic group G,

(B) if $g = g_c g_p$ is the polar decomposition of a semisimple element $g \in G$, then $g_c \in G, g_p \in G_0$.

It is also easy to see that, if $\varphi : G \to H$ is a homomorphism of algebraic linear groups and $g = g_c g_p$ is the polar decomposition of an element $g \in G$, then $\varphi(g) = \varphi(g_c)\varphi(g_p)$ is the polar decomposition of the element $\varphi(g) \in H$. The analogous property of the Jordan decomposition is well-known.

Proposition 2. *A complex linear Lie group G is algebraic if and only if (A) and (B) hold.*

Proof. Let (A) and (B) hold, and let $\bar{G}$ be the Zariski closure of G.

It is known (see, e.g., [VO]) that the group (G_0, G_0) is algebraic. Obviously, it is normal in G and hence in $\bar{G}$. Passing to the quotient $\bar{G}/(G_0, G_0)$ we may assume that G_0 is abelian.

Let $\bar{G}_0$ be the Zariski closure of G_0. We have $\bar{G}_0 = T \times U$, where T is an algebraic torus and U an abelian unipotent group. In view of (A) the subgroup $G_0 \subset \bar{G}_0$ is the direct product of its projections to T and U. Since any connected Lie subgroup of U is algebraic, we have $G_0 = T' \times U$, where T' is a connected Lie subgroup of T.

Let T_c (resp. T_p) be the real Lie subgroup of T consisting of the elements whose eigenvalues are of modulus 1 (resp. positive). Then $T = T_c \times T_p$ and, for the tangent algebras, we have $\mathfrak{t}_p = i\mathfrak{t}_c$, so T is the complex hull of the compact torus T_c. In view of (B) T' is the complex hull of a compact subtorus $T_c^{'} \subset T_c$. Since the complex hull of a compact torus is an algebraic torus, we have $T' = T$, so $G_0 = T \times U = \bar{G}_0$.

Passing to the quotient $\bar{G}/G_0$, we may assume that $G_0 = \{e\}$, i.e. G is discrete. In this case it follows from (A) and (B) that G is periodic. By a theorem of I.Schur (see, e.g., [CR]), any periodic subgroup of $\mathrm{GL}_n(C)$ is conjugate to a subgroup of U_n. If, in addition, it is discrete, it is finite. So under our assumption G is finite and hence algebraic. □

3. Proof of Theorem 1. For any $\eta \in \mathfrak{l}$ there exist $\eta_1, \eta_2, \dots \in \mathfrak{h}$ such that

$$\lim Ad(g_n)\eta_n = \eta$$

and hence

$$\lim g_n(\exp \eta_n)g_n^{-1} = \exp \eta.$$

It follows that

$$\exp \mathfrak{l} \subset L. \tag{8}$$

Let $c = c_H$ be chosen as in Proposition 1. Take any $h \in U(L, c)$. Let $h_1, h_2, \dots \in H$ be such that

$$\lim g_n h_n g_n^{-1} = h. \tag{9}$$

We may assume that $h_n \in U(H, c)$ for any n. Then $h_n = \exp \eta_n$ and hence

$$g_n h_n g_n^{-1} = \exp \mathrm{Ad}(g_n)\eta_n$$

for some $\eta_n \in U(\mathfrak{h}, c)$. In view of (9) we must have

$$\lim \mathrm{Ad}(g_n)\eta_n = \eta \in \mathfrak{l}, \quad \exp \eta = h.$$

Thus,

$$U(L, c) \subset \exp l. \tag{10}$$

It follows from (8) and (10) that L is a (complex) Lie group with tangent algebra l. To prove that it is algebraic, we are to check that (A) and (B) hold for L.

Let $h \in L$ be defined by (9). Then

$$\lim g_n(h_n)_s g_n^{-1} = h_s, \quad \lim g_n(h_n)_u g_n^{-1} = h_u,$$

so $h_s, h_u \in L$. Moreover,

$$\lim g_n(h_n)_u^t g_n^{-1} = h_u^t \in L$$

for any $t \in C$, whence $h_u \in L_0$.

The property (B) is checked in the same manner. □

4. Some invariant theory. Let us recall that, for an action of an algebraic group on an algebraic variety, any orbit is Zariski open in its Zariski closure (see, e.g., [VO]). It follows that the Zariski closure of an orbit coincides with its closure in the analytic topology. In particular, an orbit is Zariski closed if and only if it is closed in the analytic topology.

Let now a reductive algebraic group G act on an affine algebraic variety X . The algebra of polynomials on X is denoted by $C[X]$, and the subalgebra of G -invariant polynomials by $C[X]^G$. The *categorical quotient* of X with respect to the action of G , i.e. the spectrum of $C[X]^G$, is denoted by $X/\!\!/G$, and the *factorization morphism* $X \to X/\!\!/G$ defined by the embedding $C[X]^G \subset C[X]$ is denoted by π_G . A standard fact of invariant theory is that each fiber of π_G contains exactly one Zariski closed orbit. (For details see, e.g., [VP].)

Consider the action of a reductive group G on G^p by simultaneous conjugations. Denote by π_G the factorization morphism

$$\pi_G : G^p \to G^p/\!\!/G.$$

It is known [Ri] that the orbit of a p -tuple $\mathrm{g} = (g_1, \ldots, g_p) \in G^p$ is closed if and only if the subgroup $\langle \mathrm{g} \rangle = \langle g_1, \ldots, g_p \rangle$ Zariski generated by g is reductive.

For any reductive subgroup $H \subset G$ the embedding $H^p \subset G^p$ gives rise to a morphism

$$H^p/\!\!/H \to G^p/\!\!/G.$$

The main result of [Vi] is that this morphism is finite. In particular, its image $\pi_G(H^p)$ is closed in $G^p/\!\!/G$.

5. Proof of Theorem 2. First reduce the proof to the case when G is reductive. Let $\overline{G}$ be a maximal reductive subgroup of G containing H , and U the unipotent radical of G . We have

$$G = U\overline{G} \quad \text{(a semidirect product)}.$$

Let

$$g_n = u_n \overline{g}_n \quad (u_n \in U, \ \overline{g}_n \in \overline{G}).$$

Passing to a subsequence, we may assume that there exists a limit

$$\mathfrak{l}_1 = \lim \mathrm{Ad}(\overline{g}_n)\mathfrak{h}$$

and thereby a limit

$$L_1 = \lim \overline{g}_n H \overline{g}_n^{-1}.$$

Let $\overline{L}$ be the projection of L to $\overline{G}$. Obviously, $\overline{L} \subset L_1$.

Let now $\overline{S}$ be the projection of S to $\overline{G}$. Since both $\overline{S}$ and S are maximal reductive subgroups in US, they are conjugate in US and the more in G. We have

$$\overline{S} \subset \overline{L} \subset L_1,$$

so if the theorem holds for reductive groups, $\overline{S}$ is conjugate to a subgroup of H in $\overline{G}$ and, hence, S is conjugate to a subgroup of H in G.

Suppose now that G is reductive. Let $\mathbf{s} = (s_1, \ldots, s_p) \in S^p$ be a Zariski generating set of S. There exist p-tuples $\mathbf{h}_n \in H^p$ such that

$$\mathbf{s} = \lim g_n \mathbf{h}_n g_n^{-1}.$$

Applying π_G gives

$$\pi_G(\mathbf{s}) = \lim \pi_G(\mathbf{h}_n).$$

Since $\pi_G(H^p)$ is closed in $G^p /\!/ G$ (see the preceding section), we get

$$\pi_G(\mathbf{s}) \in \pi_G(H^p),$$

i.e. there exists a p-tuple $\mathbf{h} \in H^p$ such that

$$\pi_G(\mathbf{s}) = \pi_G(\mathbf{h}).$$

The fiber of π_H containing $\mathbf{h}$ contains a closed H-orbit. Replacing $\mathbf{h}$ with a representative of this orbit (lying in the same fiber of π_G), we may assume that the H-orbit of $\mathbf{h}$ itself is closed. Then the subgroup $\langle \mathbf{h} \rangle \subset H$ Zariski generated by $\mathbf{h}$ is reductive and, hence, the G-orbit of $\mathbf{h}$ is closed. Since any fiber of π_G contains only one closed orbit, it follows that the G-orbits of $\mathbf{s}$ and $\mathbf{h}$ coincide, so the subgroups Zariski generated by $\mathbf{s}$ and $\mathbf{h}$ are conjugate. Thus, S is conjugate to a subgroup of H, q.e.d.

6. Zariski dense subgroups. The following auxiliary result is needed for the proof of Theorem 3.

Proposition 3. *Any Zariski dense subgroup Γ of a connected semisimple algebraic group G contains a finitely generated Zariski dense subgroup.*

Proof. Let us first prove that Γ contains a countably generated Zariski dense subgroup. Let $\Gamma_1 \subset \Gamma$ be a countably generated subgroup whose Zariski closure $\overline{\Gamma}_1 = G_1$ has the maximal dimension. Then the connected component G_{10} of G_1 does not change if adding to Γ_1 any element of Γ. It follows that G_{10} is a normal subgroup of G. Passing to the quotient modulo G_{10}, we may assume that $G_{10} = \{e\}$. This means that any countably generated subgroup of Γ is finite, which is obviously impossible, unless $G = \{e\}$.

Let now Γ be countably generated, and let $\Gamma_2 \subset \Gamma$ be a finitely generated subgroup whose Zariski closure $\overline{\Gamma}_2 = G_2$ has the maximal dimension. As above, we reduce the proof to the case $G_{20} = \{e\}$, which means that any finitely generated subgroup of Γ is finite. In this case we are to prove that $G = \{e\}$.

We may assume $G \subset GL_n(C)$. According to a theorem of C. Jordan (see, e.g., [CR]), there exists an integer m (depending on n) such that any finite subgroup of $GL_n(C)$ contains an abelian normal subgroup of index $\leq m$, or, equivalently, admits a homomorphism with an abelian kernel to a group of order $\leq m$.

Under our assumption we have $\Gamma = \bigcup_{i=1}^{\infty} \Gamma_i$, where

$$\Gamma_1 \subset \Gamma_2 \subset \ldots$$

are finite groups. Passing to a subsequence, we may assume that each Γ_i admits a homomorphism φ_i with an abelian kernel to one and the same group Δ of order $\leq m$. Again passing to a subsequence, we may assume that any $\gamma \in \Gamma$ has one and the same image in Δ under all φ_i's for sufficiently large i . Denote this image by $\varphi(\gamma)$. In such a way we obtain a homomorphism $\varphi : \Gamma \longrightarrow \Delta$, whose kernel N is obviously abelian. Since Γ is Zariski dense in G , the Zariski closure of N is an abelian normal subgroup of G . Hence N is finite, which is impossible, unless $G = \{e\}$. □

7. Proof of Theorem 3. Suppose the conclusion of the theorem is false. Then for some neighbourhood U of e in G there exist reductive subgroups $H_1, H_2, \dots \subset G$ and p-tuples $\mathbf{h}_1, \mathbf{h}_2, \dots (\mathbf{h}_n \in H_n^p)$ such that

$$\mathbf{s} = \lim \mathbf{h}_n, \tag{11}$$

but for any n and $g \in U$

$$H_n \not\supset g S g^{-1}. \tag{12}$$

We are going to show that one may assume all H_n's to be conjugate to one and the same connected semisimple subgroup, and then to apply the theorem of Page and Richardson cited in the introduction.

Any reductive group H is a product of its connected component H_0 and some finite group (see, e.g., [Vi]). In view of this the Jordan theorem (see the preceding section) implies the existence of an integer m (depending on G) such that for any reductive subgroup $H \subset G$ the group H/H_0 admits a homomorphism with an abelian kernel to a group of order $\leq m$.

Passing to a subsequence, we may assume that for each n the group H_n/H_{n0} admits a homomorphism with an abelian kernel to one and the same group Δ of order $\leq m$. Denote by ψ_n the composition of this homomorphism and the canonical homomorphism $H_n \longrightarrow H_n/H_{n0}$. Again, passing to a subsequence, we may assume that $\psi_n(\mathbf{h}_n)$ is one and the same p-tuple $(\delta_1, \dots, \delta_p) \in \Delta^p$.

Let F be a free group on p generators and $\psi : F \longrightarrow \Delta$ the homomorphism taking the i-th generator to δ_i . Its kernel is a (normal) subgroup of finite index in F and hence finitely generated. Let $w_1, \dots, w_q$ be some generators of it. These are some words in p letters.

The subgroup generated by the elements $w_1(\mathbf{s}), \dots, w_q(\mathbf{s}) \in S$ has finite index in the subgroup Γ generated by $s_1, \dots, s_p$ and hence is Zariski dense in S . At the same time the subgroup generated by $w_1(\mathbf{h}_n), \dots, w_q(\mathbf{h}_n)$ is contained in the kernel of ψ_n . Replacing $\mathbf{s}$ with the q-tuple $(w_1(\mathbf{s}), \dots, w_q(\mathbf{s}))$ and $\mathbf{h}_n$ with the q-tuple $(w_1(\mathbf{h}_n), \dots, w_q(\mathbf{h}_n))$, we reduce the proof to the case when the group H_n/H_{n0} is abelian for any n .

Assuming this and coming back to the former notation, consider the commutator subgroup Γ' of Γ . Since Γ is Zariski dense in S , Γ' is Zariski dense in $S' = S$. By Proposition 3 Γ' contains a finitely generated subgroup Γ_1 , which is still Zariski dense in S . Let F be a free group on p generators and $w_1, \dots, w_q \in F'$ some words such that $w_1(\mathbf{s}), \dots, w_q(\mathbf{s})$ generate Γ_1 . Note that under our assumptions $w_1(\mathbf{h}_n), \dots, w_q(\mathbf{h}_n) \in H_{n0}$. Replacing $\mathbf{s}$ with the p-tuple $(w_1(\mathbf{s}), \dots, w_q(\mathbf{s}))$ and $\mathbf{h}_n$ with the p-tuple $(w_1(\mathbf{h}_n), \dots, w_q(\mathbf{h}_n))$, we reduce the proof to the case when the group H_n is connected for any n .

Repeating this trick, we reduce the proof to the case when H_n is connected and semisimple for any n.

Since there are only finitely many conjugacy classes of connected semisimple subgroups in G, we may assume that each subgroup H_n is conjugate to one and the same connected semisimple subgroup $H \subset G$. Furthermore, we may assume that there exists a limit

$$\mathfrak{l} = \lim \mathfrak{h}_n$$

and thereby a limit

$$L = \lim H_n$$

in the sense of this paper (see the introduction). By Theorem 1 L is an algebraic subgroup with tangent algebra $\mathfrak{l}$.

It follows from (11) that $L \supset S$. Hence $\mathfrak{l} \supset \mathfrak{s}$, and, for sufficiently large n, $\mathfrak{h}_n$ contains a subspace of dimension $\dim \mathfrak{s}$ arbitrarily close to $\mathfrak{s}$. By the theorem $(*)$ stated in the introduction this implies that, for sufficiently large n, $\mathfrak{h}_n$ contains a subalgebra $\mathrm{Ad}(g)\mathfrak{s}$ (and, hence, H_n contains the subgroup gSg^{-1}) with $g \in U$, which contradicts (12).

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