

NEW BOUNDS FOR LAPLACIAN ENERGY

Ivan Gutman

University of Kragujevac, Faculty of Science,

Kragujevac, Republic of Serbia,

e-mail: gutman@kg.ac.rs,

ORCID iD:  <http://orcid.org/0000-0001-9681-1550>

DOI: 10.5937/vojtehg68-24257; <https://doi.org/10.5937/vojtehg68-24257>

FIELD: Mathematics (Mathematics Subject Classification: primary 05C50, secondary 05C90)

ARTICLE TYPE: Original Scientific Paper

ARTICLE LANGUAGE: English

Abstract:

Introduction/purpose: The Laplacian energy (LE) is the sum of absolute values of the terms $\mu_i - 2m/n$, where μ_i , $i=1,2,\dots,n$, are the eigenvalues of the Laplacian matrix of the graph G with n vertices and m edges. The basic results of the theory of LE are outlined, and some new obtained.

Methods: Spectral theory of Laplacian matrices is applied.

Results: A new class of lower bounds for LE is derived.

Conclusion: The paper contributes to the Laplacian spectral theory and to the theory of graph energies.

Keywords: spectral graph theory, Laplacian spectrum (of graph), Laplacian energy.

Introduction

Throughout this paper, we are concerned with simple graphs, i.e. graphs without directed, multiple, or weighted edges, and without loops. Let G be such a graph, possessing n vertices and m edges. For details of the graph theory see (Harary, 1969), (Cvetković, 1981).

Let the vertices of the graph G be labeled by $v_1, v_2, \dots, v_n$. Let $\deg(v_i)$ be the degree (= number of first neighbors) of the vertex v_i . Then the Laplacian matrix of G , denoted by $L(G)$, is the square matrix of the order n , whose (i,j) -element is equal to -1 if the vertices v_i and v_j are adjacent, it is 0 when the vertices v_i and v_j are not adjacent, and $\deg(v_i)$ if $i=j$. The eigenvalues of $L(G)$, denoted by μ_i , $i=1,2,\dots,n$, form the Laplacian spectrum of the graph G . For details of the theory of Laplacian matrices and their spectra see (Grone et al, 1990), (Merris, 1994).

The (ordinary) *energy* of a graph is defined as the sum of absolute values of the eigenvalues of the adjacency matrix (Li et al, 2012), (Gutman & Furtula, 2019). Extending this concept to Laplacian eigenvalues, the *Laplacian energy* was defined as (Gutman & Zhou, 2006):

$$LE = \sum_{i=1}^n \left| \mu_i - \frac{2m}{n} \right|. \quad (1)$$

For details on the mathematical properties of the Laplacian energy see (Andriantiana, 2016), (Gutman & Furtula, 2019).

Preparations and the main results

The Laplacian eigenvalues μ_i , $i=1,2,\dots,n$, are non-negative real numbers. If the underlying graph G is connected, then exactly one of these eigenvalues is equal to zero (and the other $n-1$ eigenvalues are positive-valued). The following relations

$$\sum_{i=1}^n \mu_i = 2m \quad (2)$$

and

$$\sum_{i=1}^n \mu_i^2 = 2m + \sum_{i=1}^n \deg(v_i)^2 \quad (3)$$

are well known (Grone et al, 1990). At this point, we note that the sum of squares of vertex degrees is the much studied first Zagreb index; for details see (Borovičanić et al, 2017) and the references cited therein.

Combining Eqs. (2) and (3), we directly obtain

$$\sum_{i=1}^n \left| \mu_i - \frac{2m}{n} \right|^2 = 2M \quad (4)$$

where

$$M = m - \frac{2m^2}{n} + \frac{1}{2} \sum_{i=1}^n \deg(v_i)^2$$

Numerous upper and lower bounds for the Laplacian energy are known (Gutman & Zhou, 2006), (Andriantiana, 2016), of which we mention here

$$2\sqrt{M} \leq LE \leq 2M$$

and the McClelland-type upper bound

$$LE \leq \sqrt{2Mn}. \quad (5)$$

In this paper, we offer four new lower bounds for LE , namely

$$LE > \frac{abn + 2M}{a + b} \quad (6)$$

$$LE > \frac{2m}{n} + \frac{ab(n-1) + 2M - \left(\frac{2m}{n}\right)^2}{a + b} \quad (7)$$

$$LE > \frac{2\sqrt{ab}}{a + b} \sqrt{2Mn} \quad (8)$$

$$LE > \frac{2m}{n} + \frac{2\sqrt{ab}}{a + b} \sqrt{\left[2M - \left(\frac{2m}{n}\right)^2\right](n-1)} \quad (9)$$

For connected graphs with at least four vertices, all bounds (6)-(9) are strict.

The meaning of the parameters a and b is explained in the subsequent section. Observe that the multiplier in (8) and (9) is the ratio between the geometric and arithmetic means of a and b .

Proofs of bounds (6)-(9)

In order to avoid trivialities, we assume that the graphs considered are connected and have more than three vertices. Let

$$\left| \mu_i - \frac{2m}{n} \right| = X_i$$

and label the Laplacian eigenvalues of the considered graph so that

$$X_1 \geq X_2 \geq \dots \geq X_n.$$

In addition, let $X_1 = a$ and $X_n = b$. Then

$$(a - X_i)(b - X_i) = ab - (a + b)X_i + X_i^2 \leq 0 \quad (10)$$

holds for all $i=1,2,\dots,n$, and is strictly negative for at least one value of i . Summing (10) over all i , and bearing in mind Eqs. (1) and (4), we get

$$abn - (a + b)LE + 2M < 0$$

from which inequality (6) directly follows.

For connected graphs, exactly one Laplacian eigenvalue is equal to zero. Therefore, one X -value is equal to $2m/n$. Let this be $X_{\#}$.

If we sum (10) over all i , except $i = \#$, then we arrive at

$$ab(n-1) - (a+b)\left(LE - \frac{2m}{n}\right) + \left[2M - \left(\frac{2m}{n}\right)^2\right] < 0$$

which implies inequality (7).

Applying the relation $P + Q \geq 2\sqrt{PQ}$, which holds for any positive real numbers P, Q , with equality if and only if $P=Q$, we can transfer inequality (6) into inequality (8). In the very same manner, bound (9) is obtained from (7).

It is worth noting that the lower bound (8) has a similar algebraic form as the upper bound (5). Thus (5) and (8) estimate the Laplacian energy from both sides in an analogous, McClelland-type manner.

It can be shown that among the lower bounds (6)-(9), bound (7) is the best. In addition, (6) is better than (8), whereas (7) is better than (9). Numerical testing shows that bound (6) is sharper than (9). However, to verify this by exact mathematical methods seems to be a tough task and remains an open problem.

A special case: regular graphs

A graph is said to be regular if all its vertices have equal degrees. Let, thus, the considered graph G be regular, and let $\deg(v_i)=r$ for all $i=1,2,\dots,n$. Then

$$\sum_{i=1}^n \deg(v_i) = nr, \quad \sum_{i=1}^n \deg(v_i)^2 = nr^2, \quad \frac{2m}{n} = r, \quad M = m.$$

Bearing this in mind, for regular graphs, inequalities (6)-(9) reduce to:

$$LE > \frac{(ab+r)n}{a+b} \quad (6a)$$

$$LE > r + \frac{ab(n-1) + r(n-r)}{a+b} \quad (7a)$$

$$LE > \frac{2\sqrt{ab}}{a+b} n\sqrt{r} \quad (8a)$$

$$LE > r + \frac{2\sqrt{ab}}{a+b} \sqrt{r(n-r)(n-1)} \quad (9a)$$

For regular graphs, the Laplacian and ordinary energies coincide. Therefore, bounds (6a)-(9a) hold also for the ordinary energy of regular graphs. Bounds of this kind (for ordinary graph energy) were recently communicated (Oboudi, 2019), (Gutman, 2019).

References

- Andriantiana, E.O.D. 2016. Laplacian energy. In: Gutman, I. & Li, X. (Eds.), *Graph Energies - Theory and Applications*. Kragujevac: University of Kragujevac, pp.49-80.
- Borovićanin, B., Das, K.C., Furtula, B., & Gutman, I. 2017. Bounds for Zagreb indices. *MATCH Communications in Mathematical and in Computer Chemistry*, 78(1), pp.17-100 [online]. Available at: http://match.pmf.kg.ac.rs/electronic_versions/Match78/n1/match78n1_17-100.pdf. [Accessed: 30 November 2019]
- Cvetković, D. 1981. *Teorija grafova i njene primene*. Belgrade: Naučna knjiga (in Serbian).
- Grone, R., Merris, R., & Sunder, V.S. 1990. The Laplacian Spectrum of a Graph. *SIAM Journal on Matrix Analysis and Applications*, 11(2), pp.218-238. Available at: <https://doi.org/10.1137/0611016>.
- Gutman, I. 2019. Oboudi-type bounds for graph energy. *Mathematics Interdisciplinary Research*, 4(2), pp.151-155 [online]. Available at: http://mir.kashanu.ac.ir/article_96938_35e2a0e7a15dcdd1e796325b33542469.pdf. [Accessed: 30 November 2019]
- Gutman, I., & Furtula, B. 2019. *Graph Energies: Survey, Census, Bibliography*. Kragujevac: Centar SANU. Bibliography.
- Gutman, I., & Zhou, B. 2006. Laplacian energy of a graph. *Linear Algebra and its Applications*, 414(1), pp.29-37. Available at: <https://doi.org/10.1016/j.laa.2005.09.008>.
- Harary, F. 1969. *Graph Theory*. Addison-Wesley. Reading.

Li, X., Shi, Y., & Gutman, I. 2012. Introduction. In: *Graph Energy*. New York, NY: Springer Science and Business Media LLC., pp.1-9. Available at: https://doi.org/10.1007/978-1-4614-4220-2_1.

Merris, R. 1994. Laplacian matrices of graphs: A survey. *Linear Algebra and its Applications*, 197-198, pp.143-176. Available at: [https://doi.org/10.1016/0024-3795\(94\)90486-3](https://doi.org/10.1016/0024-3795(94)90486-3).

Oboudi, M.R. 2019. A new lower bound for the energy of graphs. *Linear Algebra and its Applications*, 580, pp.384-395. Available at: <https://doi.org/10.1016/j.laa.2019.06.026>.

НОВЫЕ ОГРАНИЧЕНИЯ ЛАПЛАСОВОЙ ЭНЕРГИИ

Иван Гутман

Университет в г. Крагуевац, Естественно-математический факультет,
г. Крагуевац, Республика Сербия

РУБРИКА ГРНТИ: 27.00.00 МАТЕМАТИКА

ВИД СТАТЬИ: оригинальная научная статья

ЯЗЫК СТАТЬИ: английский

Резюме:

Введение/цель: Лапласова энергия (ЛЭ) графа является суммой абсолютных величин термина $\mu_i - 2m/n$, при чем μ_i , $i=1,2,\dots,n$, представляют собственные значения матрицы Лапласа графа G с n узлами и m ветвями. Кроме основных результатов теории Лапласа, в работе приведены и некоторые нововывявленные.

Методы: В работе применялась спектральная теория матриц Лапласа.

Результаты: Выявлен новый класс предельных значений энергии Лапласа.

Выводы: Данная работа делает вклад в развитие спектральной теории Лапласа и теории энергии графов.

Ключевые слова: спектральная теория графов, лапласовский спектр (графа), энергия Лапласа.

НОВА ОГРАНИЧЕЊА ЗА ЛАПЛАСОВУ ЕНЕРГИЈУ

Иван Гутман

Универзитет у Крагујевцу, Природно-математички факултет,
Крагујевац, Република Србија

ОБЛАСТ: математика

ВРСТА ЧЛАНКА: оригинални научни рад

ЈЕЗИК ЧЛАНКА: енглески

Сажетак:

Увод/сврха: Лапласова енергија (LE) графа је сума апсолутних вредности израза $\mu_i - 2m/n$, где μ_i , $i=1,2,\dots,n$, представљају сопствене вредности Лапласове матрице графа G са n чворова и m грана. Поред основних резултата теорије Лапласове енергије дати су и неки новодобијени.

Метод: Коришћена је спектрална теорија Лапласових матрица.

Резултати: Изводи се нова класа доњих ограничења за Лапласову енергију.

Закључак: Рад даје допринос Лапласовој спектралној теорији као и теорији енергија графа.

Кључне речи: спектрална теорија графова, Лапласов спектар (графа), Лапласова енергија.

Paper received on / Дата получения работы / Датум пријема чланка: 01.12.2019.

Manuscript corrections submitted on / Дата получения исправленной версии работы / Датум достављања исправки рукописа: 20.12.2019.

Paper accepted for publishing on / Дата окончательного согласования работы / Датум коначног прихватања чланка за објављивање: 22.12.2019.

© 2020 The Author. Published by Vojnotehnički glasnik / Military Technical Courier (www.vtg.mod.gov.rs, втг.мо.упр.срб). This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution license (<http://creativecommons.org/licenses/by/3.0/rs/>).

© 2020 Автор. Опубликовано в «Военно-технический вестник / Vojnotehnički glasnik / Military Technical Courier» (www.vtg.mod.gov.rs, втг.мо.упр.срб). Данная статья в открытом доступе и распространяется в соответствии с лицензией «Creative Commons» (<http://creativecommons.org/licenses/by/3.0/rs/>).

© 2020 Аутор. Објавио Војнотехнички гласник / Vojnotehnički glasnik / Military Technical Courier (www.vtg.mod.gov.rs, втг.мо.упр.срб). Ово је чланак отвореног приступа и дистрибуира се у складу са Creative Commons licencom (<http://creativecommons.org/licenses/by/3.0/rs/>).

