MODELING AND ANALYSIS OF SINE POWER RAYLEIGH DISTRIBUTION : PROPERTIES AND APPLICATIONS

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Abstract

In this manuscript, a new probability model named as Sine Power Rayleigh distribution (SPRD) is proposed using a Sine-G function as generator. Various statistical properties of this new distribution were investigated, including the survival function, hazard function, reverse hazard rate, cumulative hazard function, mills ratio , quantile function, moments, moment generating function, conditional moments , entropy, and order statistics. The parameters of the proposed distribution were estimated using the method of maximum likelihood estimation. To assess the model's versatility and applicability, we conduct analyses on two real life data sets. The outcomes affirm the superior performance of the newly proposed model SPRD as compared to existing models .

Keywords: Sine G family, Rayleigh distribution, Sine Rayleigh distribution, Reliability Analysis, Entropy, Order Statistics, Maximum Likelihood Estimation.

1. Introduction

The concept of probability distribution has shown to be quite helpful in managing both small and large data sets. Probability distribution models are essential and widely utilised in many domains, including as physics, medicine, business management, engineering, and food. The field of probability distributions has advanced steadily due to the wide range of domains in which they are applied.Over the past few decades, researchers have used a variety of ways to introduce numerous novel probability distributions. New distributions are needed to address the problem more precisely and effectively, even though there are numerous existing ways for handling real-world data. From an applied and practical perspective, the new family of distributions modifies some of the current distributions to make them more flexible, which serves key purposes in the generalisation of distributions. There are several ways to create new models, including exponentiation, compounding, and changing and adding constants to well-known distributions.

The Rayleigh distribution (RD), named after Lord Rayleigh [15] is prominent lifetime probability model concerned with describing skewed data. The probability density function (PDF) associated with random variable x > 0 having RD with scale parameter θ is given by

$$f(x;\theta) = \frac{x}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right); \quad x > 0, \quad \theta > 0$$

and the corresponding cumulative distribution function (CDF) is given as

$$F(x;\theta) = 1 - \exp\left(-\frac{x^2}{2\theta^2}\right); \quad x > 0, \quad \theta > 0$$

In the statistical literature, numerous extensions of Rayleigh distribution (RD) have been proposed. Surless and Padgett[17] introduced the two parameter Burr type X distribution and named it as exponentiated Rayleigh distribution (ERD) or generalized Rayleigh distribution. Kundu and Raqab [11] studied and estimated the parameters of the generalized Rayleigh distribution using different estimation techniques. Ahmed et al. [2] used the square error loss function and Al-Bayyati's loss function to perform a Bayesian analysis of RD. Ajami and Jhansi [3] discussed the parameter estimation of weighted Rayleigh distribution. Ahmad et al. [1] proposed the Weibull-Rayleigh distribution and studied its characterization and parameter estimation using the transformed transformer technique. Bhat and Ahmad [6] proposed a new extension of exponentiated Rayleigh distribution and studied its various properties and demostrated its applicability by considering different datasets. Bhat and Ahmad [5] studied mathematical properties of mixture of Gamma and Rayleigh distributions. Kilai et al. [8] proposed a new versatile modification of the Rayleigh distribution for modeling COVID-19 mortality rates. Various researchers have introduced generalised distributions and their applications, see Mahmood et al. [12] , Muse et al. [13] and Ahmed et al. [15]. Bhat et al. [7] proposed a new extension of odd lindley power rayleigh distribution, studied its properties and evaluated parameter estimation techniques using both classical and Bayesian methods. Bhat and Ahmad [4] recently introduced a new generalization of the Rayleigh distribution using power transformation technique with PDF and CDF respectively given by

$$g(x;\beta,\theta) = \frac{\beta}{\theta^2} x^{2\beta-1} \exp\left(-\frac{x^{2\beta}}{2\theta^2}\right); \quad x > 0, \quad \beta,\theta > 0 \tag{1}$$

and the corresponding cumulative distribution function (CDF) is given as

$$G(x;\beta,\theta) = 1 - \exp\left(-\frac{x^{2\beta}}{2\theta^2}\right); \quad x > 0, \quad \beta,\theta > 0 \tag{2}$$

In the present manuscript, we proposed a new extension of Power Rayleigh distribution (PRD) using the Sine G family of generated distributions. The proposed distribution is named as Sine Power Rayleigh distribution (SPRD). It is more flexible and exhibits more complex shapes of density and hazard rate functions. Also, the proposed model outclass some well established models in terms of two real life data sets. The rest of the article is unfolded as : In section 2, the Ratio Transformation (RT) method is discussed. In Section 3, the PDF and CDF of the proposed model i.e., SPRD are defined. Section 4 deals with the reliability measures of the SPRD. The expansion of PDF and CDF is discussed in Section 5. Some of important statistical properties are explored in Section 6. The parameter estimation is discussed in Section 7. The simulation study and applicability of the model is debated in section 8 and 9 respectively. Finally, some conclusion are provided in Section 10.

2. SINE G FAMILY OF DISTRIBUTIONS

The CDF and PDF of the Sine G family of distributions proposed by [10] are defined by the following equations respectively:

$$F(x;\zeta) = \sin\left[\frac{\pi}{2}G(x;\zeta)\right]; \quad x \in \mathbb{R} \tag{3}$$

$$f(x;\zeta) = \frac{\pi}{2}g(x;\zeta)\cos\left[\frac{\pi}{2}G(x;\zeta)\right]; \quad x \in \mathbb{R} \tag{4}$$

Where $G(x;\zeta)$ and $g(x;\zeta)$ in equation (3) and (4) are the CDF and PDF of the base line distribution with parameter vector ζ, respectively.

3. SINE POWER RAYLEIGH DISTRIBUTION (SPRD)

The PDF of the newly proposed probability distribution Sine Power Rayleigh Distribution (SPRD)is obtained as

$$f(x;\beta,\theta) = \frac{\pi}{2}\frac{\beta}{\theta^2}x^{2\beta-1}e^{-\frac{x^{2\beta}}{2\theta^2}}\cos\left[\frac{\pi}{2}\left(1-e^{-\frac{x^{2\beta}}{2\theta^2}}\right)\right]; \quad x\in\mathbb{R}^+, \quad \beta,\theta>0 \tag{5}$$

The CDF of the newly proposed probability distribution Sine Power Rayleigh Distribution (SPRD) is obtained as

$$F(x;\beta,\theta) = \sin\left[\frac{\pi}{2}\left(1-e^{-\frac{x^{2\beta}}{2\theta^2}}\right)\right]; \quad x\in\mathbb{R}^+, \quad \beta,\theta>0 \tag{6}$$

The plots of density function of SPRD for different parameter combinations are presented in Figure 1 . It is clear from the density function plots that the proposed distribution is unimodal, decreasing, symmetric and positively skewed.

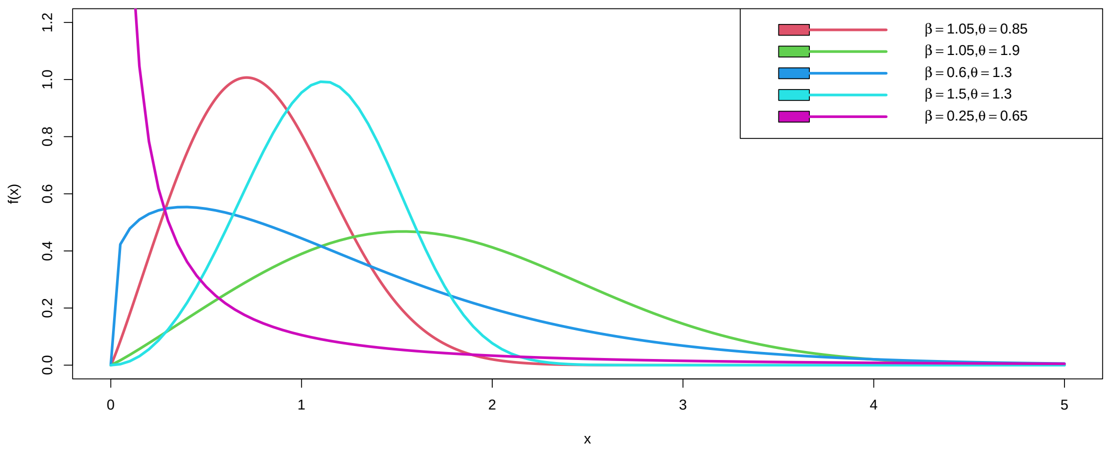

Figure 1: *Density plots of SPRD for different combinations of β and θ.*

4. RELIABILITY ANALYSIS OF THE SINE POWER RAYLEIGH DISTRIBUTION (SPRD)

This section focuses on obtaining the reliability (survival function), hazard rate (failure rate), reverse hazard function, cumulative hazard function and mills ratio expressions respectively for SPRD.

4.1. Survival function

The survival function or reliability function is defined as the probability that a system will survive beyond a specified time and is obtained for the SPRD as

$$R(x;\beta,\theta) = 1-F(x;\beta,\theta) = 1-\sin\left[\frac{\pi}{2}\left(1-e^{-\frac{x^{2\beta}}{2\theta^2}}\right)\right] \tag{7}$$

4.2. Hazard Rate

The Hazard rate evaluates a lifetime component's likelihood of failure or expiration based on the completed portion of its life, and consequently, it finds diverse applications in the analysis of lifetime distributions. Using equation (5) and (7), the expression for the hazard rate of SPRD is obtained as

$$h(x;\beta,\theta) = \frac{f(x;\beta,\theta)}{R(x;\beta,\theta)} = \frac{\frac{\pi}{2}\frac{\beta}{\theta^2}x^{2\beta-1}e^{-\frac{x^{2\beta}}{2\theta^2}}\cos\left[\frac{\pi}{2}\left(1-e^{-\frac{x^{2\beta}}{2\theta^2}}\right)\right]}{1-\sin\left[\frac{\pi}{2}(1-e^{-\frac{x^{2\beta}}{2\theta^2}})\right]} \tag{8}$$

Figure 2 depicts graphs of the hazard rate of the SPRD for different parameter values. Figure 2 suggests that the proposed distribution is quite flexible in nature and can exhibit variety of shapes such as constant, decreasing, increasing and j-shaped shaped over the parameter space.

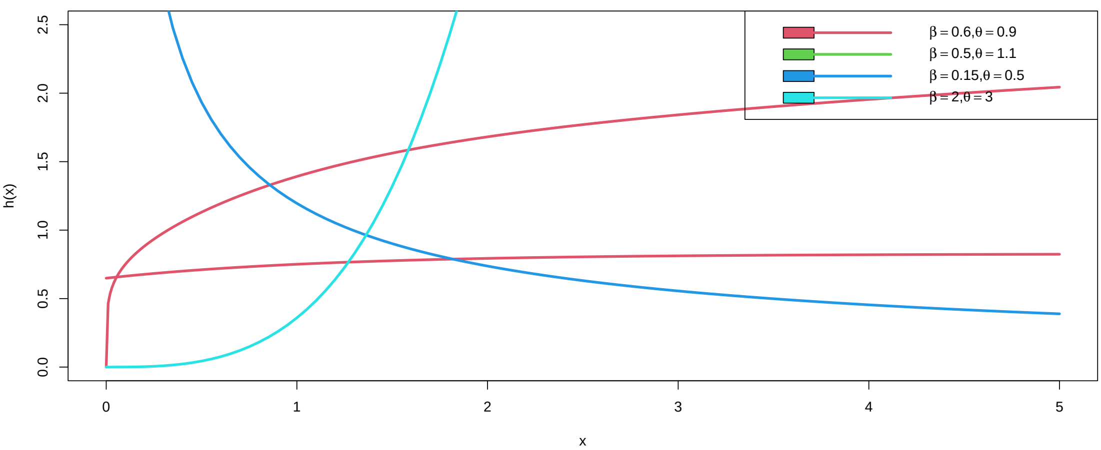

Figure 2: *Hazard rate plots of SPRD for different combinations of β and θ.*

4.3. Reverse Hazard function

The concept of reversed hazard rate of a random life is defined as the ratio between the life probability density to its distribution function . It is expressed as

$$h_r(x;\beta,\theta) = \frac{f(x;\beta,\theta)}{F(x;\beta,\theta)} = \frac{\frac{\pi}{2}\frac{\beta}{\theta^2}x^{2\beta-1}e^{-\frac{x^{2\beta}}{2\theta^2}}\cos\left[\frac{\pi}{2}\left(1-e^{-\frac{x^{2\beta}}{2\theta^2}}\right)\right]}{\sin\left[\frac{\pi}{2}\left(1-e^{-\frac{x^{2\beta}}{2\theta^2}}\right)\right]}$$

4.4. Cumulative Hazard function

The cumulative hazard function can be thought of as providing the total accumulated risk of experiencing the event of interest that has been gained by progressing to time t. The cumulative hazard function for the SPRD is defined as

$$\Lambda_{SPRD}(x;\beta,\theta) = -\log R(x;\beta,\theta) = -\log\left\{1-\sin\left[\frac{\pi}{2}\left(1-e^{-\frac{x^{2\beta}}{2\theta^2}}\right)\right]\right\}$$

4.5. Mills Ratio

The mills ratio for the SPRD is defined as

$$M.R = \frac{F(x;\beta,\theta)}{R(x;\beta,\theta)} = \frac{\sin\left[\frac{\pi}{2}\left(1-e^{-\frac{x^{2\beta}}{2\theta^2}}\right)\right]}{1-\sin\left[\frac{\pi}{2}\left(1-e^{-\frac{x^{2\beta}}{2\theta^2}}\right)\right]} \tag{9}$$

4.6. Quantile function

The quantile function for the SPRD is given by

$$x = \left[-2\theta^2 log\left(1-\frac{2}{\pi}\sin^{-1}u\right)\right]^{\frac{1}{2\beta}} \tag{10}$$

The first quartile (Q_1), median (Q_2), and third quartile (Q_3) can be derived by setting $u = \frac{1}{4}, \frac{1}{2}$, and $\frac{3}{4}$ in equation (10) respectively.

5. Expansion of PDF and CDF

Various statistical properties can be easily deduced by using mixture representation of PDF and CDF of the proposed model.
expansion of $\cos\left[\frac{\pi}{2}\left(1-e^{-\frac{x^{2\beta}}{2\theta^2}}\right)\right]$ can be expressed as

$$\cos\left[\frac{\pi}{2}\left(1-e^{-\frac{x^{2\beta}}{2\theta^2}}\right)\right] = \sum_{l=0}^{\infty}\frac{(-1)^l}{2l!}\frac{\pi^{2l}}{2^{2l}}\left(1-e^{-\frac{x^{2\beta}}{2\theta^2}}\right)^{2l}$$

Also $\left(1-e^{-\frac{x^{2\beta}}{2\theta^2}}\right)^{2l}$ can be expressed as

$$\left(1-e^{-\frac{x^{2\beta}}{2\theta^2}}\right)^{2l} = \sum_{m=0}^{\infty}(-1)^m\binom{2l}{m}e^{-\frac{mx^{2\beta}}{2\theta^2}}$$

expansion of $\sin\left[\frac{\pi}{2}\left(1-e^{-\frac{x^{2\beta}}{2\theta^2}}\right)\right]$ can be expressed as

$$\sin\left[\frac{\pi}{2}\left(1-e^{-\frac{x^{2\beta}}{2\theta^2}}\right)\right] = \sum_{p=0}^{\infty}\frac{(-1)^p}{(2p+1)!}\frac{\pi^{2p+1}}{2^{2p+1}}\left(1-e^{-\frac{x^{2\beta}}{2\theta^2}}\right)^{2p+1}$$

Also $\left(1-e^{-\frac{x^{2\beta}}{2\theta^2}}\right)^{2p+1}$ can be expressed as

$$\left(1-e^{-\frac{x^{2\beta}}{2\theta^2}}\right)^{2p+1} = \sum_{q=0}^{\infty}(-1)^q\binom{2p+1}{q}e^{-\frac{qx^{2\beta}}{2\theta^2}}$$

Thus, the PDF and CDF of the proposed model can be written in the mixture representation respectively as

$$f(x;\beta,\theta) = \frac{\beta}{\theta^2}x^{2\beta-1}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{l+m}}{2l!}\binom{2l}{m}\frac{\pi^{2l+1}}{2^{2l+1}}e^{-\frac{(m+1)x^{2\beta}}{2\theta^2}} \tag{11}$$

$$F(x;\beta,\theta) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-1)^{p+q}}{(2p+1)!}\binom{2p+1}{q}\frac{\pi^{2p+1}}{2^{2p+1}}e^{-\frac{qx^{2\beta}}{2\theta^2}} \tag{12}$$

6. STATISTICAL PROPERTIES OF SPRD

Some of the mathematical properties such as the r^{th} moment, moment generating function, conditional moments and associated measures, the entropy and order statistics are derived.

6.1. Moments

The r^{th} moment of the SPRD can be evaluated directly by extending the PDF given in equation (11)

$$E(X^r) = \int_0^\infty x^r f(x;\beta,\theta)dx \quad ,r = 1,2,..$$

where f(x) is the PDF of the SPRD given in equation (11), thus

$$E(X^r) = \frac{\beta}{\theta^2}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{l+m}}{2l!}\binom{2l}{m}\frac{\pi^{2l+1}}{2^{2l+1}}\int_0^\infty x^{r+2\beta-1}e^{-\frac{(m+1)x^{2\beta}}{2\theta^2}} \tag{13}$$

Using integration via substitution method in equation (13), we perform the following operations.

let $\frac{(m+1)x^{2\beta}}{2\theta^2} = z \implies x = (\frac{2\theta^2 z}{m+1})^{\frac{1}{2\beta}}$, such that $dx = \frac{1}{2\beta}(\frac{2\theta^2}{m+1})^{\frac{1}{2\beta}}(z)^{\frac{1}{2\beta}-1}$

Thus, simplifyingequation (13) yields

$$E(X^r) = (2\theta^2)^{\frac{r}{2\beta}}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{l+m}}{2l!}\binom{2l}{m}\frac{\pi^{2l+1}}{2^{2l+1}}\left(\frac{1}{m+1}\right)^{\frac{r}{2\beta}+1}\Gamma(\frac{r}{2\beta}+1) \tag{14}$$

where,

$\Gamma(\frac{r}{2\beta}+1) = \int_0^\infty z^{(\frac{r}{2\beta}+1)-1}e^{-z}dz$

setting $r = 1$ in equation (14) the mean of the model is computed as

$$E(X) = (2\theta^2)^{\frac{1}{2\beta}}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{l+m}}{2l!}\binom{2l}{m}\frac{\pi^{2l+1}}{2^{2l+1}}\left(\frac{1}{m+1}\right)^{\frac{1}{2\beta}+1}\Gamma(\frac{1}{2\beta}+1) \tag{15}$$

Similarly for r = 2, 3 and 4 in equation (14) ,the second, third and fourth moment about origin are respectively calculated as

$$E(X^2) = (2\theta^2)^{\frac{1}{\beta}}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{l+m}}{2l!}\binom{2l}{m}\frac{\pi^{2l+1}}{2^{2l+1}}\left(\frac{1}{m+1}\right)^{\frac{1}{\beta}+1}\Gamma(\frac{1}{\beta}+1) \tag{16}$$

$$E(X^3) = (2\theta^2)^{\frac{3}{2\beta}}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{l+m}}{2l!}\binom{2l}{m}\frac{\pi^{2l+1}}{2^{2l+1}}\left(\frac{1}{m+1}\right)^{\frac{3}{2\beta}+1}\Gamma(\frac{3}{2\beta}+1) \tag{17}$$

$$E(X^4) = (2\theta^2)^{\frac{2}{\beta}}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{l+m}}{2l!}\binom{2l}{m}\frac{\pi^{2l+1}}{2^{2l+1}}\left(\frac{1}{m+1}\right)^{\frac{2}{\beta}+1}\Gamma(\frac{2}{\beta}+1) \tag{18}$$

6.2. Moment Generating function of SPRD

we can calculate moment generating function based on the r^{th} moment of SPRD as given by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) \tag{19}$$

$$M_X(t) = (2\theta^2)^{\frac{r}{2\beta}} \sum_{r=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{t^r}{r!} \frac{(-1)^{l+m}}{2l!} \binom{2l}{m} \frac{\pi^{2l+1}}{2^{2l+1}} \left(\frac{1}{m+1}\right)^{\frac{r}{2\beta}+1} \Gamma(\frac{r}{2\beta}+1) \tag{20}$$

6.3. Conditional moments and associated measures

In this section, the expression for conditional moments is acquired. But first we will introduce an important lemma which will be applied in the next section.

Lemma 1. Let us suppose a random variable X follows SPRD (β, θ) with PDF given in equation (11) and let $\varphi_r(z) = \int_0^z x^r f(x;\beta,\theta)\, dx$ denotes the rth incomplete moment, then we have

$$\varphi_r(z) = (2\theta^2)^{\frac{r}{2\beta}} \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{l+m}}{2l!} \binom{2l}{m} \frac{\pi^{2l+1}}{2^{2l+1}} \left(\frac{1}{m+1}\right)^{\frac{r}{2\beta}+1} \gamma\left(\left(\frac{r}{2\beta}+1\right), \frac{(m+1)z^{2\beta}}{2\theta^2}\right) \tag{21}$$

where $\gamma(a,b) = \int\limits_0^b z^{a-1}e^{-z}dz$ denotes the lower incomplete gamma function.

Proof: Using the PDF of SPRD given in equation (11) , we have

$$\varphi_r(z) = \int\limits_0^z x^r f(x;\beta,\theta)dx = \frac{\beta}{\theta^2} \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{l+m}}{2l!} \binom{2l}{m} \frac{\pi^{2l+1}}{2^{2l+1}} \int\limits_0^z x^{r+2\beta-1} e^{-\frac{(m+1)x^{2\beta}}{2\theta^2}} \tag{22}$$

On Simplification, we obtain

$$\varphi_r(z) = (2\theta^2)^{\frac{r}{2\beta}} \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{l+m}}{2l!} \binom{2l}{m} \frac{\pi^{2l+1}}{2^{2l+1}} \left(\frac{1}{m+1}\right)^{\frac{r}{2\beta}+1} \gamma\left(\left(\frac{r}{2\beta}+1\right), \frac{(m+1)z^{2\beta}}{2\theta^2}\right) \tag{23}$$

Setting r=1 in equation (23) will yield first incomplete moment as given by

$$\varphi_1(z) = (2\theta^2)^{\frac{1}{2\beta}} \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{l+m}}{2l!} \binom{2l}{m} \frac{\pi^{2l+1}}{2^{2l+1}} \left(\frac{1}{m+1}\right)^{\frac{1}{2\beta}+1} \gamma\left(\left(\frac{1}{2\beta}+1\right), \frac{(m+1)z^{2\beta}}{2\theta^2}\right) \tag{24}$$

6.3.1 Lorenz and Bonferroni inequality Curves

The Lorenz and Bonferroni inequality curves are an important application of the first incomplete moment. For a given probability distribution, they are defined by

$$L_p = \frac{1}{E(X)} \int_0^t x f(x;\beta,\theta)\, dx = \frac{\varphi_1(t)}{E(X)}$$

$$L_p = \frac{\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{l+m}}{2l!}\binom{2l}{m}\frac{\pi^{2l+1}}{2^{2l+1}}\left(\frac{1}{m+1}\right)^{\frac{1}{2\beta}+1}\gamma\left(\left(\frac{1}{2\beta}+1\right),\frac{(m+1)t^{2\beta}}{2\theta^2}\right)}{\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{l+m}}{2l!}\binom{2l}{m}\frac{\pi^{2l+1}}{2^{2l+1}}\left(\frac{1}{m+1}\right)^{\frac{1}{2\beta}+1}\Gamma(\frac{1}{2\beta}+1)}$$

Similarly,

$$B_P = \frac{1}{pE(X)}\int_0^t xf(x;\beta,\theta)\,dx = \frac{\varphi_1(t)}{pE(X)}$$

$$B_P = \frac{\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{l+m}}{2l!}\binom{2l}{m}\frac{\pi^{2l+1}}{2^{2l+1}}\left(\frac{1}{m+1}\right)^{\frac{1}{2\beta}+1}\gamma\left(\left(\frac{1}{2\beta}+1\right),\frac{(m+1)t^{2\beta}}{2\theta^2}\right)}{p\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{l+m}}{2l!}\binom{2l}{m}\frac{\pi^{2l+1}}{2^{2l+1}}\left(\frac{1}{m+1}\right)^{\frac{1}{2\beta}+1}\Gamma(\frac{1}{2\beta}+1)}$$

6.3.2 r^{th} Conditional Moment and r^{th} Reversed Conditional Moment of SPRD

The r^{th} conditional moment of the SPRD is calculated by

$$E\left[X^r|x>t\right] = \frac{1}{R(t)}\int_t^{\infty} x^r f(x;\beta,\theta)\,dx = \frac{1}{R(t)}\left[E(X^r)-\varphi_r(t)\right]$$

where $R(t)$ is the reliability of SPRD at time t.
Inserting the value of equation (7), (14) and (23), we obtain

$$E\left[X^r|x>t\right] = \frac{\left(2\theta^2\right)^{\frac{r}{2\beta}}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{l+m}}{2l!}\binom{2l}{m}\frac{\pi^{2l+1}}{2^{2l+1}}\left(\frac{1}{m+1}\right)^{\frac{r}{2\beta}+1}\left[\Gamma\left(\frac{r}{2\beta}+1\right)-\gamma\left(\left(\frac{r}{2\beta}+1\right),\frac{(m+1)t^{2\beta}}{2\theta^2}\right)\right]}{1-\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-1)^{p+q}}{(2p+1)!}\binom{2p+1}{q}\frac{\pi^{2p+1}}{2^{2p+1}}e^{-\frac{qt^{2\beta}}{2\theta^2}}}$$

Similarly, the r^{th} reversed conditional moment of the SPRD is defined by

$$E\left[X^r|x\le t\right] = \frac{1}{F(t)}\int_0^t x^r f(x;\beta,\theta)\,dx = \frac{\varphi_r(t)}{F(t)}$$

$$E\left[X^r|x\le t\right] = \frac{\left(2\theta^2\right)^{\frac{r}{2\beta}}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{l+m}}{2l!}\binom{2l}{m}\frac{\pi^{2l+1}}{2^{2l+1}}\left(\frac{1}{m+1}\right)^{\frac{r}{2\beta}+1}\gamma\left(\left(\frac{r}{2\beta}+1\right),\frac{(m+1)t^{2\beta}}{2\theta^2}\right)}{\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-1)^{p+q}}{(2p+1)!}\binom{2p+1}{q}\frac{\pi^{2p+1}}{2^{2p+1}}e^{-\frac{qt^{2\beta}}{2\theta^2}}}$$

6.3.3 Mean Residual Life (MRL) and Mean Waiting Time (MWT)

The MRL is defined as

$$\mu(t) = \frac{1}{R(t)}\left[E(t)-\int_0^t xf(x;\beta,\theta)\,dx\right]-t = \frac{1}{R(t)}\left[E(t)-\varphi_1(t)\right]-t$$

After inserting the value of equation (7), (15) and (24), we obtain the required expression for mean residual life as

$$\mu(t) = \frac{\left(2\theta^2\right)^{\frac{1}{2\beta}}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{l+m}}{2l!}\binom{2l}{m}\frac{\pi^{2l+1}}{2^{2l+1}}\left(\frac{1}{m+1}\right)^{\frac{1}{2\beta}+1}\left[\Gamma\left(\frac{1}{2\beta}+1\right)-\gamma\left(\left(\frac{1}{2\beta}+1\right),\frac{(m+1)t^{2\beta}}{2\theta^2}\right)\right]}{1-\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-1)^{p+q}}{(2p+1)!}\binom{2p+1}{q}\frac{\pi^{2p+1}}{2^{2p+1}}e^{-\frac{qt^{2\beta}}{2\theta^2}}}-t$$

The MWT is defined as

$$\overline{\mu}(t) = t - \frac{1}{F(t)} \int_0^t x f(x;\beta,\theta)\, dx = t - \frac{\varphi_1(t)}{F(t)}$$

$$\overline{\mu}(t) = t - \frac{(2\theta^2)^{\frac{1}{2\beta}} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{l+m}}{2l!} \binom{2l}{m} \frac{\pi^{2l+1}}{2^{2l+1}} \left(\frac{1}{m+1}\right)^{\frac{1}{2\beta}+1} \gamma\left(\left(\frac{1}{2\beta}+1\right), \frac{(m+1)t^{2\beta}}{2\theta^2}\right)}{\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(-1)^{p+q}}{(2p+1)!} \binom{2p+1}{q} \frac{\pi^{2p+1}}{2^{2p+1}} e^{-\frac{qt^{2\beta}}{2\theta^2}}}$$

6.4. Renyi entropy

The entropy of a random variable is defined as the average amount of information lost during a random experiment. The Renyi entropy, which Alfred Renyi introduced [16] and generalises Shannon's measure of information, is defined as

$$R_\eta = \frac{1}{1-\eta} \log \int_{-\infty}^{\infty} f^\eta(x;\beta,\theta)\, dx, \quad \eta > 0, \quad \eta \neq 1$$

Using the PDF given in equation (11), we have

$$R_\eta = \frac{1}{1-\eta} \log \left(\frac{\beta}{\theta}\right)^\eta \left(\sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{l+m}}{2l!} \binom{2l}{m} \frac{\pi^{2l+1}}{2^{2l+1}}\right)^\eta \int_0^\infty x^{\eta(2\beta-1)} e^{-\frac{\eta(m+1)x^{2\beta}}{2\theta^2}}$$

$$R_\eta = \frac{1}{1-\eta} \log \left(\frac{\beta}{\theta}\right)^\eta \frac{1}{2\beta} \left(\sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{l+m}}{2l!} \binom{2l}{m} \frac{\pi^{2l+1}}{2^{2l+1}}\right)^\eta \left(\frac{2\theta^2}{\eta(m+1)}\right)^{\frac{\eta(2\beta-1)+1}{2\beta}} \Gamma\left(\frac{\eta(2\beta-1)+1}{2\beta}\right)$$

6.5. Order Statistics of SPRD

The order statistics connected to the SPRD is devoted in this section. Let $x_{(r;n)}$ be the r^{th} order statistics with the random sample $x_{(1)}, x_{(2)}, x_{(3)}, ...x_{(n)}$ derived from the SPRD having the PDF $f(X;,\beta,\theta)$ and CDF $F(X;\beta,\theta)$. Therefore, the PDF and CDF of $x_{(r;n)}$ say $f_{(r;n)}(x)$ and $F_{(r;n)}(x)$ are respectively defined as

$$f_{(r;n)}(x) = \frac{1}{B(n, n-r+1)} \left[F(x;\beta,\theta)\right]^{r-1} \left[1 - F(x;\beta,\theta)\right]^{n-r} f(x;\beta,\theta) \tag{25}$$

$$F_{(r;n)}(x) = \sum_{j=r}^{n} \binom{n}{j} \left[F(x;\beta,\theta)\right]^j \left[1 - F(x;\beta,\theta)\right]^{n-j} \tag{26}$$

Using equation (5) and equation (6) in equation (25) and equation (26), the PDF and CDF of r^{th} ordered statistics for the SPRD are derived and are expressed as

$$f_{(r;n)}(x) = \frac{\frac{\pi}{2} \frac{\beta}{\theta^2} x^{2\beta-1} e^{-\frac{x^{2\beta}}{2\theta^2}} \cos\left[\frac{\pi}{2}\left(1 - e^{-\frac{x^{2\beta}}{2\theta^2}}\right)\right]}{B(n, n-r+1)} \left\{\sin\left[\frac{\pi}{2}\left(1 - e^{-\frac{x^{2\beta}}{2\theta^2}}\right)\right]\right\}^{r-1} \left\{1 - \sin\left[\frac{\pi}{2}\left(1 - e^{-\frac{x^{2\beta}}{2\theta^2}}\right)\right]\right\}^{n-r}$$

$$F_{(r;n)}(x) = \sum_{j=r}^{n} \binom{n}{j} \left\{ \sin\left[\frac{\pi}{2}\left(1-e^{-\frac{x^{2\beta}}{2\theta^2}}\right)\right]\right\}^{j} \left\{1-\sin\left[\frac{\pi}{2}\left(1-e^{-\frac{x^{2\beta}}{2\theta^2}}\right)\right]\right\}^{n-j}$$

where $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the beta function.

7. Estimation of Parameters

The goal of this study is to estimate the unknown parameters β and θ of the SPRD using Maximum Likelihood Estimation (MLE). we assume that x_1, x_2, ..., x_n be a random sample of n observations drawn from the SPRD (β, θ) with unknown parametric vector $\Theta = (\beta, \theta)^T$.

7.1. Maximum Likelihood Estimation (MLE)

Here, Maximum Likelihood Estimation (MLE) approach is used to obtain the estimators of the unknown parameters of SPRD (β, θ) . The likelihood function is given by

$$L(\Theta) = \left[\frac{\pi\beta}{2\theta^2}\right]^n e^{-\sum_{i=1}^{n}\frac{x_k^{2\beta}}{2\theta^2}} \prod_{k=1}^{n} x_k^{2\beta-1} \cos\left[\frac{\pi}{2}(1-e^{-\frac{x_k^{2\beta}}{2\theta^2}})\right]$$

For the parametric vector $(\Theta) = (\beta, \theta)^T$, the logarithm likelihood function is expressed as

$$\ell = n\log\left(\frac{\pi}{2}\right) + n\log(\beta) - 2n\log(\theta) - \frac{1}{2\theta^2}\sum_{k=1}^{n} x_k^{2\beta} + (2\beta-1)\sum_{k=1}^{n}\log x_k + \sum_{k=1}^{n}\log\cos\left[\frac{\pi}{2}\left(1-e^{-\frac{x_k^{2\beta}}{2\theta^2}}\right)\right] \quad (27)$$

The elements of the score vector $U(\Theta)$ =(U_β, U_θ) are obtained by partially differentiating Equation (27) with respect to the model parameters and are given by

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} + 2\sum_{k=1}^{n}\ln(x_k) - \frac{1}{2\theta^2}\sum_{k=1}^{n} x_k^{2\beta}\ln(x_k) - \frac{\pi}{4\theta^2}\sum_{k=1}^{n}\tan\left[\frac{\pi}{2}\left(1-e^{-\frac{x_k^{2\beta}}{2\theta^2}}\right)\right] e^{-\frac{x_k^{2\beta}}{2\theta^2}} x_k^{2\beta}\ln(x_k)$$

$$\frac{\partial\ell}{\partial\theta} = \frac{-2n}{\theta} + \frac{1}{\theta^3}\sum_{k=1}^{n} x_k^{2\beta} + \frac{\pi}{2\theta^3}\sum_{k=1}^{n}\tan\left[\frac{\pi}{2}\left(1-e^{-\frac{x_k^{2\beta}}{2\theta^2}}\right)\right] e^{-\frac{x_k^{2\beta}}{2\theta^2}} x_k^{2\beta}$$

The likelihood estimates of the model parameters can be obtained by setting the score vector U($\boldsymbol{\Theta}$) = 0 . Since, the above equations are non-linear and hence the model parameters are estimated using Newton-Raphson algorithm.

8. SIMULATION ILLUSTRATION

In this section, we carry out simulation study using R software to examine the behaviour of MLE's for various sample sizes.We generate the random samples of size $25, 75, 150, 300$ and 500 from

SPRD and repeat the process for 1000 times in R software.Various combinations of parameters are chosen as $(1.5, 1.35)$ and $(0.5, 2.2)$ with relation to the standard order (β, θ). The average MLE values, bias, and related empirical mean squared errors (MSEs) were determined for each scenario. Tables 1 exhibits the ML estimates, bias and MSE. We observe from table 1 that the agreement between theory and practice improves as the sample size n increases. MSE and bias of the estimators suggest that the estimators are consistent and the maximum likelihood estimator of the parameters perform quite well and the results are precise and accurate. The MSE decreases with increasing sample size under all conditions.

Table 1: *MLE,Bias and MSE for the parameters β and θ*

sample size	Parameters		MLE		Bias		MSE	
n	β	θ	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\theta}$
25	1.5	1.35	1.58963	1.38116	0.21193	0.15622	0.07685	0.04192
75			1.52863	1.36292	0.11586	0.08563	0.02170	0.01211
150			1.51474	1.35744	0.07911	0.05752	0.00999	0.00543
300			1.50528	1.35236	0.05462	0.03945	0.00459	0.00248
500			1.50487	1.35130	0.04267	0.03108	0.00278	0.00153
25	0.5	2.2	0.53233	2.39244	0.07177	0.40725	0.00960	0.36239
75			0.50767	2.24504	0.03767	0.20412	0.00222	0.06987
150			0.50439	2.22579	0.02799	0.14628	0.00126	0.03659
300			0.50299	2.21458	0.01852	0.10195	0.00054	0.01658
500			0.50085	2.20587	0.01432	0.07776	0.00034	0.00967

9. APPLICATION

This section is devoted to illustrate the flexibility, adaptability, and suitability of the SPRD, by means of two real data sets . We compare the proposed distribution with the following models :

- Power Rayleigh distribution (PRD) With PDF given as

$$f(x; \beta, \theta) = \frac{\beta}{\theta^2} x^{2\beta-1} \exp\left(-\frac{x^{2\beta}}{2\theta^2}\right); \quad \beta, \theta > 0$$

- Weighted Rayleigh Distribution (WRD) with PDF given as

$$f(x; \beta, \theta) = \frac{x^{\beta+1} \exp\left(-\frac{x^2}{2\theta^2}\right)}{\theta^{\beta+2} 2^{\frac{\beta}{2}} \Gamma\left(\frac{\beta}{2}+1\right)}; \quad \beta, \theta > 0$$

- Rayleigh distribution (RD) with PDF given as

$$f(x; \theta) = \frac{x}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right); \quad \theta > 0$$

Here, several goodness of fit criterion such as -2ll, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Akaike Information Criterion Corrected (AICC) , Kolmogorov-Smirnov (KS) and P value statistics are used. The statistic with the lowest value of -2ll, AIC, BIC, AICC,K-S and maximum value of P value is considered the best fit.

9.1. Data Set 1

Data set 1:The first data is on the breaking stress of carbon fibres of 50 mm length (GPa). The data has been previously used by [4] and [14] . The data is as follows:
0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90

Table 2: *Estimates (standard errors), -2ll, AIC, BIC, AICC, K-S statistic and P-value for Data-set 1.*

Model	$\hat{\beta}$	$\hat{\theta}$	$-2ll$	AIC	BIC	AICC	K-S	P-value
SPRD	1.6366 (0.1595)	5.8515 (1.2057)	171.6825	175.6825	180.0618	175.8730	0.0791	0.8029
PRD	1.7205 (0.1654)	4.8502 (1.0369)	172.1352	176.1352	180.5145	176.3256	0.0823	0.7625
WRD	2.5727 (0.7452)	1.3551 (0.1234)	175.7107	179.7107	184.0900	179.9012	0.1104	0.3963
RD		2.0491 (0.1261)	196.4168	198.4168	200.6065	198.4793	0.2265	0.0022

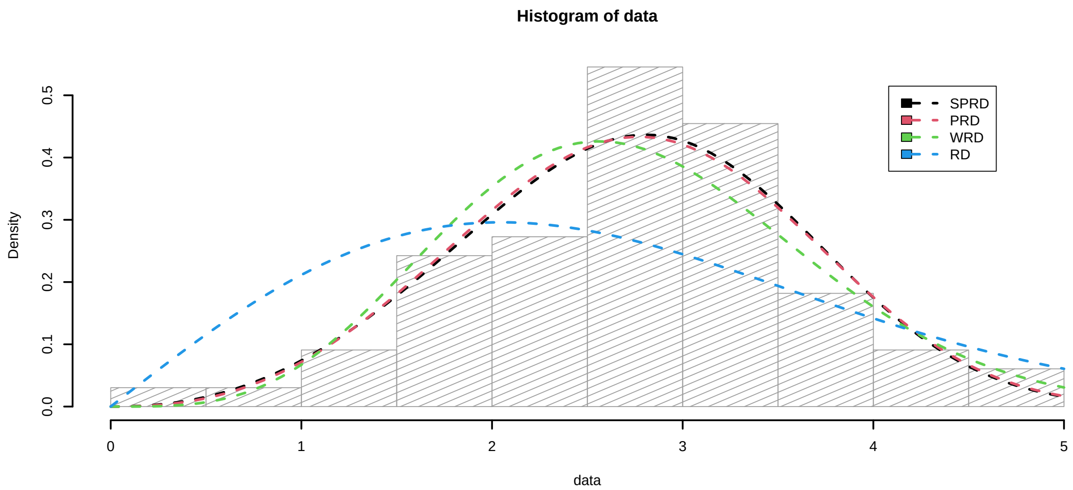

Figure 3: *Fitted density plots for dataset 1*

9.2. Data set 2

Data set 2: Consider the following data set in Johnson and Kotz [9] and represent the survival times (in years) after diagnosis of 43 patients with a certain kind of leukemia.
0.019, 0.129, 0.159, 0.203, 0.485, 0.636, 0.748, 0.781, 0.869,1.175, 1.206, 1.219, 1.219, 1.282, 1.356, 1.362, 1.458, 1.564, 1.586, 1.592, 1.781, 1.923, 1.959, 2.134, 2.413, 2.466, 2.548, 2.652, 2.951, 3.038, 3.6, 3.655, 3.745, 4.203, 4.690, 4.888,5.143, 5.167, 5.603, 5.633, 6.192, 6.655, 6.874

Table 3: *Estimates (standard errors), -2ll, AIC, BIC, AICC,K-S statistic and P-value for Data-set 2.*

Model	$\hat{\beta}$	$\hat{\theta}$	$-2ll$	AIC	BIC	AICC	K-S	P-value
SPRD	0.5887 (0.0736)	1.6864 (0.2041)	162.9906	166.9906	170.5130	167.2906	0.0869	0.901
PRD	0.6198 (0.0766)	1.3094 (0.1647)	163.2203	167.2203	170.7427	167.5203	0.0903	0.8744
WRD	0.0010 (0.3799)	2.2409 (0.2728)	181.9592	185.9592	189.4816	186.2592	0.2423	0.0128
RD		2.2415 (0.1709)	181.9277	183.9277	185.6889	184.0252	0.2421	0.0128

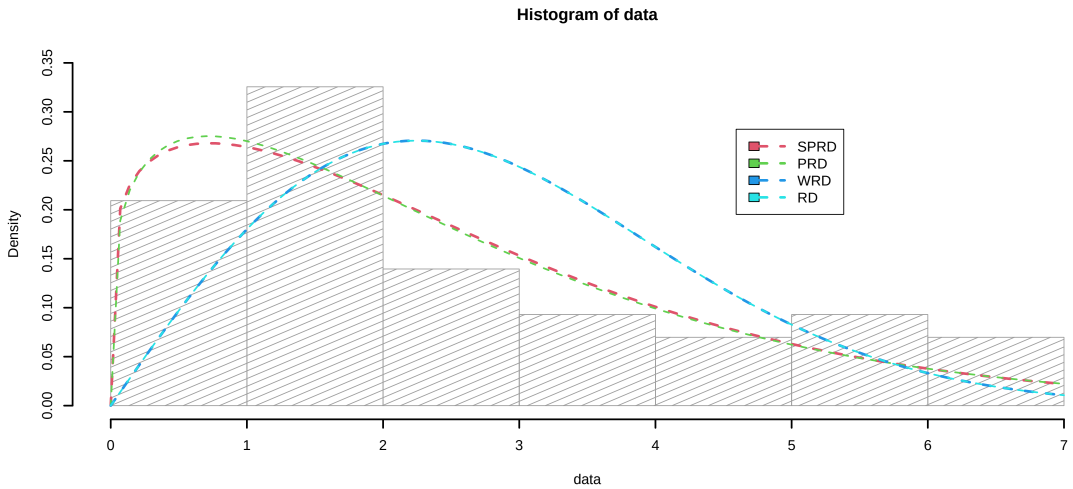

Figure 4: *Fitted density plots for dataset 2*

The results obtained in Table 2 and Table 3 reveal that SPRD has the least value of all the comparison criterions, hence SPRD can be considered a strong competitor to other distributions compared here for fitting data. The plots of the fitted models are displayed in figure 3 and 4. Also, from these plots , it is evident that SPRD provides a close fit to the two data sets.

10. CONCLUSION

In this paper, a new life time distribution namely Sine Power Rayleigh distribution (SPRD) is proposed and studied. The SPRD model is an expansion that incorporates the Sine-G family of distributions introduced by [10] resulting in a novel trigonometric distribution. The new distribution is more flexible and its hazard rate function exhibits complex shapes. The study derives various properties of the proposed distribution, including the survival function, hazard rate function, reverse hazard function, cumulative hazard function, moments, moment generating function, quantile function, Lorenz and Bonferroni inequality curves, Renyi entropy and order statistics.The parameters of the proposed distribution are estimated using the maximum likelihood method and a simulation study is conducted to assess the performance of the maximum likelihood estimators (MLEs) for these parameters. Furthermore, the effectiveness of the proposed distribution is evaluated by applying it to two distinct real life datasets and comparing it with well known standard distributions such as the Rayleigh distribution, Power Rayleigh distribution and Weighted Rayleigh distribution. The results demonstrate that the Sine Power Rayleigh distribution (SPRD) surpasses its competitors in terms of fitting the two datasets.

References

[1] A. Ahmad, S. P. Ahmad, and A. Ahmed. Characterization and estimation of weibull-rayleigh distribution with applications to life time data. *Appl. Math. Inf. Sci. Lett*, 5:71–79, 2017.

[2] A. Ahmed, S. P. Ahmad, and J. Reshi. Bayesian analysis of rayleigh distribution. *International Journal of Scientific and Research Publications*, 3(10):1–9, 2013.

[3] M. Ajami and S. Jahanshahi. Parameter estimation in weighted rayleigh distribution. *Journal of Modern Applied Statistical Methods*, 16(2):14, 2017.

[4] A. A. Bhat and S. P. Ahmad. A new generalization of rayleigh distribution: Properties and applications. *Pakistan journal of statistics*, 36(3), 2020.

[5] A. A. Bhat and S. P. Ahmad. Mixture of gamma and rayleigh distributions: Mathematical properties and applications. *Journal of Applied Probability*, 16(2):81–97, 2021.

[6] A. A. Bhat and S. P. Ahmad. An extension of exponentiated rayleigh distribution: Properties and applications. *Thailand Statistician*, 21(1):209–227, 2023.

[7] A. A. Bhat, S. P. Ahmad, E. M. Almetwally, N. Yehia, N. Alsadat, and A. H. Tolba. The odd lindley power rayleigh distribution: properties, classical and bayesian estimation with applications. *Scientific African*, 20:e01736, 2023.

[8] M. Kilai, G. A. Waititu, W. A. Kibira, M. Abd El-Raouf, and T. A. Abushal. A new versatile modification of the rayleigh distribution for modeling covid-19 mortality rates. *Results in Physics*, 35:105260, 2022.

[9] S. Kotz, N. Balakrishnan, C. B. Read, and B. Vidakovic. *Encyclopedia of Statistical Sciences, Volume 1*. John Wiley & Sons, 2005.

[10] D. Kumar, U. Singh, and S. K. Singh. A new distribution using sine function-its application to bladder cancer patients data. *Journal of Statistics Applications & Probability*, 4(3):417, 2015.

[11] D. Kundu and M. Z. Raqab. Generalized rayleigh distribution: different methods of estimations. *Computational statistics & data analysis*, 49(1):187–200, 2005.

[12] Z. Mahmood, T. M Jawa, N. Sayed-Ahmed, E. Khalil, A. H. Muse, A. H. Tolba, et al. An extended cosine generalized family of distributions for reliability modeling: Characteristics and applications with simulation study. *Mathematical Problems in Engineering*, 2022, 2022.

[13] A. H. Muse, A. H. Tolba, E. Fayad, O. A. Abu Ali, M. Nagy, M. Yusuf, et al. Modelling the covid-19 mortality rate with a new versatile modification of the log-logistic distribution. *Computational Intelligence and Neuroscience*, 2021, 2021.

[14] S. U. Rasool and S. P. Ahmad. Power length biased weighted lomax distribution. *Reliability: Theory & Applications*, 17(4 (71)):543–558, 2022.

[15] L. Rayleigh. Xii. on the resultant of a large number of vibrations of the same pitch and of arbitrary phase. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 10(60):73–78, 1880.

[16] A. Rényi. On measures of entropy and information. In *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics*, volume 4, pages 547–562. University of California Press, 1961.

[17] J. Surles and W. Padgett. Inference for reliability and stress-strength for a scaled burr type x distribution. *Lifetime data analysis*, 7:187–200, 2001.