UDC 530.1; 539.1

QUANTUM BILLIARDS WITH BRANES

V. D. Ivashchuk [a,b], **V. N. Melnikov** [a,b]

[a] Center for Gravitation and Fundamental Metrology, VNIIMS, Ozyornaya st., 46, Moscow 119361, Russia.
[b] Institute of Gravitation and Cosmology, Peoples' Friendship University of Russia, Miklukho-Maklaya St., 6, Moscow 117198, Russia.

E-mail: ivashchuk@mail.ru

Cosmological Bianchi-I type model in the $(n+1)$-dimensional gravitational theory with several forms is considered. When electric non-composite brane ansatz is adopted the Wheeler-DeWitt (WDW) equation for the model, written in the conformally-covariant form, is analyzed. Under certain restrictions asymptotic solutions to WDW equation near the singularity are found which reduce the problem to the so-called quantum billiard on the $(n-1)$-dimensional Lobachevsky space H^{n-1}.

Keywords: *cosmological billiards, branes, Wheeler-DeWitt equation.*

1 Introduction

In this paper we deal with the quantum billiard approach for multidimensional cosmological-type models defined on the manifold $(u_-, u_+) \times \mathbb{R}^n$, where $n \geq 3$. In classical case the billiard approach was suggested by Chitre [1] for explanation the BLK-oscillations [2] in the Bianchi-IX model [3] by using a simple triangle billiard in the Lobachevsky space H^2.

In multidimensional case the billiard representation for cosmological model with multicomponent "perfect" fluid was introduced in [4,5]. The billiard approach for multidimensional models with scalar fields and fields of forms was suggested in [6], see also [7] for examples of "chaotic" behavior in supergravitational models.

Recently the quantum billiard approach for a multidimensional gravitational model with several forms was considered in [8]. The asymptotic solutions to WDW equation presented in [8] are equivalent to those obtained earlier in [5].

Here we use another form of the WDW equation with enlarged minisuperspace which include the form potentials Φ^s [9]. We get another version of the quantum billiard approach, which is different from that of [8].

2 The model

Here we consider the multidimensional gravitational model governed by the action

$$S_{act} = \frac{1}{2\kappa^2} \int_M d^D z \sqrt{|g|} \{R[g] - \sum_{s \in S} \frac{\theta_s}{n_s!} (F^s)^2\} + S_{YGH}, \qquad (1)$$

where $g = g_{MN} dz^M \otimes dz^N$ is the metric on the manifold M, $\dim M = D$, $\theta_s \neq 0$, $F^s = dA^s = \frac{1}{n_s!} F^s_{M_1 \dots M_{n_a}} dz^{M_1} \wedge \ldots \wedge dz^{M_{n_s}}$ is a n_s-form ($n_s \geq 2$) on a D-dimensional manifold M, $s \in S$. In (1) we denote $|g| = |\det(g_{MN})|$, $(F^s)^2 = F^s_{M_1 \dots M_{n_s}} F^s_{N_1 \dots N_{n_a}} g^{M_1 N_1} \dots g^{M_{n_s} N_{n_s}}$, $s \in S$, where S is some finite set of indices and $S_{\rm YGH}$ is the standard York-Gibbons-Hawking boundary term.

Let us consider the manifold $M = \mathbb{R}_* \times \mathbb{R}^n$ with the metric

$$g = w e^{2\gamma(u)} du \otimes du + \sum_{i=1}^{n} e^{2\phi^i(u)} \varepsilon(i) dx^i \otimes dx^i, \qquad (2)$$

where $\mathbb{R}_* = (u_-, u_+)$, $w = \pm 1$ and $\varepsilon(i) = \pm 1$, $i = 1, \dots, n$. The dimension of M is $D = 1 + n$. For $w = -1$ and $\varepsilon(i) = 1$, $i = 1, \dots, n$, we deal with cosmological solutions while for $w = 1$, and $\varepsilon(1) = -1$, $\varepsilon(j) = 1$, $j = 2, \dots, n$, we get static solutions (e.g. wormholes etc).

Let $\Omega = \Omega(n)$ be a set of all non-empty subsets of $\{1, \dots, n\}$. For any $I = \{i_1, \dots, i_k\} \in \Omega$, $i_1 < \ldots < i_k$, we denote $\tau(I) \equiv dx^{i_1} \wedge \ldots \wedge dx^{i_k}$, $\varepsilon(I) \equiv \varepsilon(i_1) \dots \varepsilon(i_k)$, $d(I) = |I| \equiv k$.

For the fields of forms we consider the following non-composite electric ansatz

$$A^s = \Phi^s \tau(I_s), \qquad F^s = d\Phi^s \wedge \tau(I_s), \qquad (3)$$

where $\Phi^s = \Phi^s(u)$ is smooth function on $\mathbb{R}_*$ and $I_s \in \Omega$, $s \in S$. Due to (3) we have $d(I_s) = n_s - 1$, $s \in S$.

The equations of motion for the model (1) with the fields from (2) and (3) are equivalent to equations of motion for the σ-model governed by the action [9]

$$S_\sigma = \frac{\mu}{2} \int du \mathcal{N} \left\{ \mathcal{G}_{AB}(X) \dot{X}^A \dot{X}^B \right\}, \qquad (4)$$

where $\mu \neq 0$ and $\mathcal{N} = \exp(\gamma_0 - \gamma) > 0$ is modified lapse function with $\gamma_0(\phi) \equiv \sum_{i=1}^n \phi^i$, $X = (X^A) =$

$(\phi^i, \Phi^s) \in \mathbb{R}^N$, $N = n + m$, $m = |S|$ is the number of branes, $\dot{X} \equiv dX/du$ and minisupermetric $\mathcal{G} = \mathcal{G}_{AB}(X) dX^A \otimes dX^B$ on minisuperspace $\mathcal{M} = \mathbf{R}^N$ is defined by the relation

$$\mathcal{G} = G + \sum_{s \in S} \varepsilon_s e^{-2U^s(\phi)} d\Phi^s \otimes d\Phi^s, \tag{5}$$

where

$$G = G_{ij} d\phi^i \otimes d\phi^j, \qquad G_{ij} = \delta_{ij} - 1, \tag{6}$$

and

$$U^s(\phi) = U^s_i \phi^i = \sum_{i \in I_s} \phi^i, \quad U^s = (U^s_i) = \delta_{iI_s}, \tag{7}$$

$s \in S$.

Here $\delta_{iI} = \sum_{j \in I} \delta_{ij}$ is an indicator of i belonging to I: $\delta_{iI} = 1$ for $i \in I$ and $\delta_{iI} = 0$ otherwise; and $\varepsilon_s = \varepsilon(I_s)\theta_s$, $s \in S$.

In what follows we will use the scalar product

$$(U, U') = G^{ij} U_i U'_j, \tag{8}$$

for $U = (U_i), U' = (U'_i) \in \mathbb{R}^n$, where (G^{ij}) is the matrix inverse to the matrix (G_{ij}) $G^{ij} = \delta^{ij} + \frac{1}{2-D}$, $i, j = 1, \ldots, n$.

3 Quantum billiard approach

First we outline two restrictions which will be used in derivation of the quantum billiard: (i) $d(I_s) < D-2$, (ii) $\varepsilon_s > 0$, for all s.

Due to the first restriction we get

$$(U^s, U^s) > 0, \quad s \in S. \tag{9}$$

Let us fix the temporal gauge as follows

$$\gamma_0 - \gamma = 2f(X), \quad \mathcal{N} = e^{2f}, \tag{10}$$

where $f \colon \mathcal{M} \to \mathbf{R}$ is a smooth function. Then we obtain the Lagrange system with the Lagrangian

$$L_f = \frac{\mu}{2} e^{2f} \mathcal{G}_{AB}(X) \dot{X}^A \dot{X}^B \tag{11}$$

and the energy constraint

$$E_f = \frac{\mu}{2} e^{2f} \mathcal{G}_{AB}(X) \dot{X}^A \dot{X}^B = 0. \tag{12}$$

Using the standard prescriptions of covariant and conformally covariant quantization of the energy constraint [10] we are led to the Wheeler-DeWitt (WDW) equation [9]

$$\hat{H}^f \Psi^f \equiv \left(-\frac{1}{2\mu} \Delta \left[e^{2f} \mathcal{G} \right] + \frac{a}{\mu} R \left[e^{2f} \mathcal{G} \right] \right) \Psi^f = 0, \tag{13}$$

where

$$a = a_N = \frac{(N-2)}{8(N-1)}, \tag{14}$$

$N = n + m$.

Here $\Psi^f = \Psi^f(X)$ is the wave function corresponding to the f-gauge (10) and satisfying the relation

$$\Psi^f = e^{bf} \Psi^{f=0}, \quad b = (2-N)/2. \tag{15}$$

In (13) we denote by $\Delta[\mathcal{G}^f]$ and $R[\mathcal{G}^f]$ the Laplace-Beltrami operator and the scalar curvature corresponding to the metric

$$\mathcal{G}^f = e^{2f} \mathcal{G}, \tag{16}$$

respectively.

The metrics G, $\mathcal{G}$ have pseudo-Euclidean signatures $(-,+,...,+)$. We put

$$e^{2f} = -(G_{ij} \phi^i \phi^j)^{-1}, \tag{17}$$

where $G_{ij} \phi^i \phi^j < 0$.

In what follows we will use a diagonalization of ϕ-variables

$$\phi^i = S^i_a z^a, \tag{18}$$

$a = 0, ..., n-1$, obeying $G_{ij} \phi^i \phi^j = \eta_{ab} z^a z^b$, where $(\eta_{ab}) = \mathrm{diag}(-1, +1, ..., +1)$.

We restrict the WDW equation to the lower light cone $V_- = \{ z = (z^0, \vec{z}) | z^0 < 0, \eta_{ab} z^a z^b < 0 \}$ and introduce Misner-Chitre-like coordinates

$$z^0 = -e^{-y^0} \frac{1 + \vec{y}^2}{1 - \vec{y}^2}, \tag{19}$$

$$\vec{z} = -2e^{-y^0} \frac{\vec{y}}{1 - \vec{y}^2}, \tag{20}$$

where $y^0 < 0$ and $\vec{y}^2 < 1$. We note that in these variables $f = y^0$.

We denote

$$\bar{G}_{ij} = e^{2f} G_{ij}, \qquad \bar{G}^{ij} = e^{-2f} G^{ij}. \tag{21}$$

The following formula is valid

$$\bar{G} = -dy^0 \otimes dy^0 + h_L, \tag{22}$$

where

$$h_L = \frac{4 \delta_{rs} dy^r \otimes dy^s}{(1 - \vec{y}^2)^2}. \tag{23}$$

Here the metric h_L is defined on the unit ball $D^{n-1} = \{ \vec{y} \in \mathbb{R}^{n-1} | \vec{y}^2 < 1 \}$. The pair (D^{n-1}, h_L) is one of the realization of $(n-1)$-dimensional analogue of the Lobachevsky space.

We use the following ansatz

$$\Psi^f = e^{C(\phi)} e^{iQ_s \Phi^s} \Psi_{0,L}, \tag{24}$$

where

$$C(\phi) = \frac{1}{2}(\sum_{s \in S} U_i^s \phi^i - mf). \tag{25}$$

Here parameters $Q_s \neq 0$ correspond to charge densities of branes and $e^{iQ_s \Phi^s} = \exp(i \sum_{s \in S} Q_s \Phi^s)$.

Then the WDW is reduced to the following relation

$$\left(-\frac{1}{2}\Delta[\bar{G}] + \frac{1}{2}\sum_{s \in S} Q_s^2 e^{-2f + 2U^s(\phi)} + \delta V \right) \times \Psi_{0,L} = 0, \tag{26}$$

where

$$\delta V = A e^{-2f} - \frac{1}{8}(n-2)^2 \tag{27}$$

and

$$A = \frac{1}{8(N-1)}[\sum_{s,s' \in S} (U^s, U^{s'}) -(N-2)\sum_{s \in S}(U^s, U^s)]. \tag{28}$$

It was shown in [6] that

$$\frac{1}{2}\sum_{s \in S} Q_s^2 e^{-2f + 2U^s(\phi)} \to V_\infty, \tag{29}$$

as $y^0 = f \to -\infty$.

In this relation V_∞ is the potential of infinite walls which are produced by branes:

$$V_\infty = \sum_{s \in S} \theta_\infty(\vec{v}_s^2 - 1 - (\vec{y} - \vec{v}_s)^2). \tag{30}$$

Here we use the notation $\theta_\infty(x) = +\infty$ for $x \geq 0$ and $\theta_\infty(x) = 0$ for $x < 0$. The vectors $\vec{v}_s$, $s \in S$, belonging to $\mathbb{R}^{n-1}$ are defined by the formulae

$$\vec{v}_s = -\vec{u}_s / u_{s0}, \tag{31}$$

where n-dimensional vectors $u_s = (u_{s0}, \vec{u}_s) = (u_{sa})$ are obtained from U^s-vectors using a diagonalization matrix (S_a^i) from (18)

$$u_{sa} = S_a^i U_i^s. \tag{32}$$

Due to condition (9)

$$(U^s, U^s) = -(u_{s0})^2 + (\vec{u}_s)^2 > 0 \tag{33}$$

for all s. Here we use a diagonalization (18) from [6] obeying

$$u_{s0} > 0 \tag{34}$$

for all $s \in S$. The inverse matrix $(S_i^a) = (S_a^i)^{-1}$ defines the the map inverse to (18)

$$z^a = S_i^a \phi^i, \tag{35}$$

$a = 0, ..., n-1$.

The inequalities (33) imply $|\vec{v}_s| > 1$ for all s. The potential V_∞ corresponds to the billiard B in the multidimensional Lobachevsky space (D^{n-1}, h_L). This billiard is an open domain in D^{n-1} which is defined by a set of inequalities:

$$|\vec{y} - \vec{v}_s| < \sqrt{\vec{v}_s^2 - 1} = r_s, \tag{36}$$

$s \in S$. The boundary ∂B is formed by parts of hyperspheres with centers in $\vec{v}_s$ and radii r_s.

The condition (34) is also obeyed for the diagonalization (35) with

$$z^0 = U_i \phi^i / \sqrt{|(U,U)|}, \tag{37}$$

where U-vector is time-like $(U,U) < 0$ and $(U, U^s) < 0$ for all $s \in S$.

Thus, we are led to an asymptotic relation for the function $\Psi_{0,L}(y^0, \vec{y})$

$$\left(-\frac{1}{2}\Delta[\bar{G}] + \delta V \right) \Psi_{0,L} = 0 \tag{38}$$

with $\vec{y} \in B$ and the zero boundary condition $\Psi_{0,L|\partial B} = 0$ imposed. Due to (22) we get $\Delta[\bar{G}] = -(\partial_0)^2 + \Delta[h_L]$, where $\Delta[h_L] = \Delta_L$ is the Laplace-Beltrami operator corresponding to the $(n-1)$-dimensional Lobachevsky metric h_L.

By splitting the variables

$$\Psi_{0,L} = \Psi_0(y^0)\Psi_L(\vec{y}) \tag{39}$$

we are led to the asymptotic relation (for $y^0 \to -\infty$)

$$\left(\left(\frac{\partial}{\partial y^0}\right)^2 - \Delta_L + 2Ae^{-2y^0} + E - \tfrac{1}{4}(n-2)^2 \right) \times \Psi_0 = 0 \tag{40}$$

equipped with the relations

$$\Delta_L \Psi_L = -E \Psi_L, \qquad \Psi_{L|\partial B} = 0. \tag{41}$$

Here we assume that the operator $(-\Delta_L)$ with the zero boundary condition imposed has a spectrum obeying

$$E \geq \frac{1}{4}(n-2)^2. \tag{42}$$

This inequality was proved in [8] for billiards with finite volumes.

Here we put

$$A < 0. \tag{43}$$

Solving equation (40) we get for $A < 0$ the following basis of solutions

$$\Psi_0 = \mathcal{B}_{i\omega}\left(\sqrt{2|A|}e^{-y^0}\right), \tag{44}$$

where $\mathcal{B}_{i\omega}(z) = I_{i\omega}(z), K_{i\omega}(z)$ are modified Bessel functions and

$$\omega = \sqrt{E - \frac{1}{4}(n-2)^2} \geq 0. \tag{45}$$

It was shown in [11] that

$$\Psi^f \to 0 \tag{46}$$

as $y^0 \to -\infty$ for fixed $\vec{y} \in B$ and $\Phi^s \in \mathbb{R}$, $s \in S$, in the following two cases: i) $\mathcal{B} = K$; ii) $\mathcal{B} = I$, when $\frac{1}{2}q > \sqrt{2|A|}$.

In [11] we have presented an example of quantum $d = 9$ billiard for $D = 11$ gravitational model with 120 "electric" 4-forms and have shown the asymptotic vanishing of the basis wave functions $\Psi^f \to 0$, as $y^0 \to -\infty$, for any choice of the Bessel function $\mathcal{B} = K, I$. The generalization of the model to electromagnetic composite case (when scalar fields were present) was done in [12].

4 Conclusion

Here we have done an overview of our approach from [11, 12] by considering the quantum billiard for the cosmological-type model with n one-dimensional factor-spaces in the theory with several forms. After adopting the electric non-composite brane ansatz with certain restrictions on parameters of the model we have deduced the Wheeler-DeWitt (WDW) equation for the model, written in the conformally-covariant form.

By imposing certain restrictions on parameters of the model we have obtained the asymptotic solutions to WDW equation which are of a quantum billiard form since they are governed by the spectrum of the Laplace-Beltrami operator on the billiard with the zero boundary condition imposed. The billiard is a part of the $(n-1)$-dimensional Lobachevsky space H^{n-1}.

Acknowledgement

This research has been supported in part by PFUR grant (No. 200312-1-174) in 2014.

References

[1] Chitre D. M. 1972 Ph. D. Thesis (University of Maryland).

[2] Belinskii V. A., Lifshitz E. M. and Khalatnikov I. M. 1970 *Usp. Fiz. Nauk* **102** 463 [in Russian].

[3] Misner C. W. 1969 *Phys. Rev.* **186** 1319.

[4] Ivashchuk V. D., Kirillov A. A. and Melnikov V. N. 1994 *Russian Physics Journal* **37** 1102.

[5] Ivashchuk V. D. and Melnikov V. N. 1995 *Class. Quantum Grav.* **12** 809.

[6] Ivashchuk V. D. and Melnikov V. N. 2000 *J. Math. Phys.* **41** 634.

[7] Damour T., Henneaux M. and Nicolai H. 2003 **20** R145.

[8] Kleinschmidt A., Koehn M. and Nicolai H. 2009 *Phys. Rev.* **D 80** 061701.

[9] Ivashchuk V. D. and Melnikov V. N. 1998 *J. Math. Phys.* **39** 2866.

[10] Misner C. W. 1972 In *Magic without Magic. John Archibald Wheeler, a collection of essays in honor of his sixtieth birthday* ed. Klauder J. R. (Freeman, San Francisko).

[11] Ivashchuk V. D. and Melnikov V. N. 2013 *Grav. Cosmol.* **19** 171.

[12] Ivashchuk V. D. and Melnikov V. N. 2014 *Eur. Phys. J. C* **74** 2805.

Received 13.11.2014

В. Д. Иващук, В. Н. Мельников

КВАНТОВЫЕ БИЛЬЯРДЫ С БРАНАМИ

Рассмотрена космологическая модель типа Бианки-I в $(n+1)$-мерной гравитационной теории с несколькими полями форм. В случае, когда принят анзатц с электрическими некомпозитными бранами, проанализировано уравнение Уилера-ДеВитта (УДВ), записанное в конформно-ковариантном виде. При определенных ограничениях найдены асимптотические решения уравнения УДВ вблизи сингулярности, которые сводят проблему к так называемому квантовому бильярду на $(n-1)$-мерном пространстве Лобачевского H^{n-1}.

Ключевые слова: *космологические бильярды, браны, уравнение Уилера-ДеВитта.*

Иващук В. Д., доктор физико-математических наук, ведущий научный сотрудник.
ВНИИ метрологической службы.
Ул. Озёрная, 46, 119361 Москва, Россия.
E-mail: ivashchuk@mail.ru

Мельников В. Н., доктор физико-математических наук, профессор, главный научный сотрудник.
ВНИИ метрологической службы.
Ул. Озёрная, 46, 119361 Москва, Россия.
E-mail: melnikov@phys.msu.ru