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Article

A uniqueness theorem for mean periodic functions on the Bessel – Kingmann hypergroup

G. V. Krasnoschekikh✉, **Vit. V. Volchkov**

Donetsk State University, 24 Universitetskaya St., Donetsk 283001, Russia

Gleb V. Krasnoschekikh, wolverimred@mail.ru, https://orcid.org/0009-0005-2783-4333
Vitaliy V. Volchkov, volna936@gmail.com, https://orcid.org/0000-0003-4274-0034, SPIN: 4478-1677, AuthorID: 7006247848

Abstract. One of the properties of a periodic function on the real axis is that it is completely determined by its values on the period. This fact admits the following nontrivial multidimensional generalization: if a function $f \in C^\infty(\mathbb{R}^n)$ $(n \geqslant 2)$ with zero integrals over all spheres (or balls) of fixed radius r is zero in some ball of radius r, then f is zero in $\mathbb{R}^n$. The condition of infinite smoothness of the function f in this statement cannot be relaxed. In this paper, we study a similar phenomenon for solutions of convolution equations related to the generalized Bessel shift operator. First, we consider the case when the convolution factor in the equation is an indicator of a segment symmetric with respect to zero. It is shown that the solutions to such an equation are determined by their values on the specified segment. Further, a generalization of this property for the general Bessel convolution equation is given. The results obtained are analogues of the well-known uniqueness theorems for mean periodic functions belonging to F. John, Yu.I. Lyubich and A.F. Leontiev.
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Теорема единственности для периодических в среднем функций на гипергруппе Бесселя – Кингмана

Г. В. Краснощеких✉, **Вит. В. Волчков**

Донецкий государственный университет, Россия, 283001, г. Донецк, ул. Университетская, д. 24

Краснощеких Глеб Витальевич, аспирант кафедры математического анализа и дифференциальных уравнений, wolverimred@mail.ru, https://orcid.org/0009-0005-2783-4333
Волчков Виталий Владимирович, заведующий кафедрой математического анализа и дифференциальных уравнений, volna936@gmail.com, https://orcid.org/0000-0003-4274-0034, SPIN: 4478-1677, AuthorID: 505219

Аннотация. Одно из свойств периодической функции на вещественной оси состоит в том, что она полностью определяется своими значениями на периоде. Этот факт допускает следующее нетривиальное обобщение на многомерный случай: если функция $f \in C^{\infty}(\mathbb{R}^n)$ $(n \geqslant 2)$ с нулевыми интегралами по всем сферам (или шарам) фиксированного радиуса r равна нулю в некотором шаре радиуса r, то f является нулевой на $\mathbb{R}^n$. Условие бесконечной гладкости функции f в этом утверждении ослабить нельзя. В данной работе изучается подобное явление для решений уравнений свертки, связанных с оператором обобщенного сдвига Бесселя. Сначала рассматривается случай, когда свертывателем уравнения является индикатор отрезка, симметричного относительно нуля. Показано, что решения такого уравнения определяется своими значениями на указанном отрезке. Далее приводится обобщение этого свойства для общего уравнения свертки Бесселя. Полученные результаты являются аналогами известных теорем единственности для периодических в среднем функций, принадлежащих Ф. Йону, Ю. И. Любичу и А. Ф. Леонтьеву.

Ключевые слова: обобщенный сдвиг, уравнения свертки, функции Бесселя, сферическое преобразование, теорема Титчмарша о свертке

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Introduction

The well-known uniqueness theorem of F. John (see [1, Chap. 6; 2]) states that if a function $f \in C^{\infty}(\mathbb{R}^n)$ $(n \geqslant 2)$ with zero integrals over all spheres (or balls) of fixed radius r is zero in some ball of radius r, then f is zero in $\mathbb{R}^n$. The condition of infinite smoothness of the function f in this statement cannot be relaxed (see [1, Chap. 6] for $n = 2, 3$ and [3, Theorems 14.7 and 14.9] for the general case).

The theorem of F. John has been further developed and refined in different directions. In particular, spectral analogues of F. John's theorem were established [3, Theorem 14.1], its generalizations and refinements were studied for the twisted spherical means [4], for weighted spherical means on a sphere [5], on symmetric spaces and the Heisenberg group [3, Chap. 15–17], and for solutions of convolution equations [6, Theorem 1; 7, 8; 9, Theorem 8; 10, Chap. 5; 11–13].

In addition to their independent interest, the results obtained turned out to be important due to their applications in extreme problems of integral geometry, the theory of lacunary series, the support problem, the theory of harmonic functions as well as in the study of various classes of mean periodic functions and their generalizations (see [14]). Furthermore, it was found that F. John's theorem and its analogues have deep connections with microlocal analysis, which is widely used in research on partial differential equations [15, 16].

Various issues of analysis and differential equations related to the generalized translation operator were studied by J. Delsart, B. M. Levitan, K. Trimèche, I. A. Kipriyanov, S. M. Sitnik, S. S. Platonov, etc. (see [17–21] and the bibliography in these works). In [22], a study of the injectivity of the spherical mean operator on the Chébli-Trimèche hypergroups was initiated and a local two-radii theorem was proved. In this paper, we obtain an analogue of John's uniqueness theorem for mean periodic functions on the Bessel – Kingmann hypergroup, which is a model representative of the above-mentioned class of hypergroups.

In addition to self-interest, the study of spherical means and their generalizations on hypergroups is important due to the following circumstances. The fact is that various properties of mean periodic functions on multidimensional spaces allowed us to obtain earlier applications to other issues of analysis only for a certain discrete set of parameters depending on the dimension of the space, the multiplicity of root subspaces of the Lie algebra of the isometry group, etc. Establishing the appropriate results on hypergroups will make it possible to remove this restriction.

It should be noted that general information on the theory of hypergroups and hypercomplex systems is contained in the review by G. L. Litvinov [23] and the monograph by Yu. M. Berezansky and A. A. Kalyuzhny [24]. Our attention to the study of mean periodic functions on hypergroups was attracted by A. A. Kalyuzhny during the reports of the second author at the seminar of Yu. M. Berezansky and M. L. Gorbachuk (Institute of Mathematics, Kiev, 2009, 2010). The authors thank A. A. Kalyuzhny and the seminar participants for their interest and useful discussions.

1. Statement of the main result

Let H be an arbitrary set, and let Φ be a linear space of complex-valued functions defined on H. Suppose that each element $x \in H$ is associated with a linear operator R^x in Φ, and for any fixed $y \in H$ the function $\psi(x) = R^x\varphi(y)$ is contained in Φ for all $\varphi \in \Phi$. A set H is called a *hypergroup* if the following conditions (see [23, Sect. 2; 24, Chap. 1, Sect. 2]) are satisfied:

1) for any elements $x, y \in H$ the relation

$$R^x L^y = L^y R^x$$

is valid, where

$$L^y\varphi(x) = R^x\varphi(y), \quad \varphi \in \Phi;$$

2) there exists an element $e \in H$ such that $R^e = I$ (I is the identity operator).

In this case, it is said that the operators R^x form a family of *generalized translation operators*.

The Bessel – Kingmann hypergroup corresponds to the case when $H = \mathbb{R}_+ = [0, +\infty)$, $\Phi = C([0, +\infty))$ and $R^x = T_x^\alpha$, where $\alpha > -1/2$,

$$T_x^\alpha f(y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int\limits_0^\pi f(\sqrt{x^2+y^2-2xy\cos\theta})(\sin\theta)^{2\alpha} d\theta, \quad x, y \geqslant 0. \tag{1}$$

It is convenient to assume that f is extended on $\mathbb{R}$ to an even function and

$$T_x^\alpha f(y) = T_{|x|}^\alpha f(|y|), \quad x, y \in \mathbb{R}.$$

We denote by $L_{\natural,\alpha}^{1,\mathrm{loc}}(I_R)$ the space of even locally summable functions with respect to the measure $|x|^{2\alpha+1}dx$ on the interval $I_R = (-R, R)$, $0 < R \leqslant +\infty$. In this paper, we study an equation of the form

$$\int\limits_0^r T_y^\alpha f(x) x^{2\alpha+1} dx = 0, \quad y \in I_{R-r} \quad (0 < r < R), \tag{2}$$

where $f \in L_{\natural,\alpha}^{1,\mathrm{loc}}(I_R)$. The main result of this paper is the following uniqueness theorem for solutions of equation (2).

Theorem 1. *Let $0 < r < R \leqslant +\infty$. Suppose that a function $f \in L_{\natural,\alpha}^{1,\mathrm{loc}}(I_R)$ satisfies condition* (2) *and $f = 0$ on I_r. Then $f = 0$ on I_R.*

Thus, the solutions of equation (2) of class $L_{\natural,\alpha}^{1,\mathrm{loc}}(I_R)$ are completely determined by their values on the interval I_r. Similar results in $\mathbb{R}^n$ and other multidimensional spaces (see [3, Chaps. 14–17; 14, Part 2, Chaps. 1–3]) require additional conditions on the smoothness of the function f in the neighborhood of its null set. Note also that the size of the null set I_r in Theorem 1 cannot be reduced in general (see [14, Part 2, Sect. 1.2, Theorem 1.2]).

2. Auxiliary assertions

Throughout the following, we assume that α is a fixed number from the interval $(-1/2, +\infty)$. As usual, the symbols $\mathbb{N}$ and $\mathbb{Z}_+$ denote the sets of natural and non-negative integers respectively.

Let $0 < r < R \leqslant +\infty$, $\overline{I}_r = [-r, r]$. If $f, g \in L^{1,\mathrm{loc}}_{\natural,\alpha}(I_R)$ and $\operatorname{supp} g \subset \overline{I}_r$ ($\operatorname{supp} g$ is the support of the function g), then the Bessel (Hankel) convolution

$$(f \star g)(y) = \int_0^r (T_y^\alpha f)(x) g(x) x^{2\alpha+1} dx, \quad y \in I_{R-r} \tag{3}$$

is defined, which belongs to the class $L^{1,\mathrm{loc}}_{\natural,\alpha}(I_{R-r})$. In particular, if $g = \chi_r$ is the indicator of the segment $\overline{I}_r$, then

$$(f \star \chi_r)(y) = \int_0^r (T_y^\alpha f)(x) x^{2\alpha+1} dx, \quad y \in I_{R-r}.$$

Thus, to study equation (2) it is convenient to introduce the classes

$$V_r(I_R) = \{f \in \ L^{1,\mathrm{loc}}_{\natural,\alpha}(I_R) : f \star \chi_r = 0 \text{ on } I_{R-r}\},$$

$$V_r^m(I_R) = V_r(I_R) \cap C_\natural^m(I_R), \quad m \in \mathbb{Z}_+ \cup \{\infty\},$$

where $C_\natural^m(I_R)$ is the space of even m times continuously differentiable functions on I_R.

The convolution (3) naturally extends to even distributions f, g at least one of which has a compact support. The main properties of this convolution are contained in [18, Chap. 6; 21, Sect. 2; 22, Sect. 2]. For example, the operator $f \to f \star g$ commutes with the Bessel differential operator, i.e.

$$L_\alpha(f \star g) = (L_\alpha f) \star g = f \star (L_\alpha g), \tag{4}$$

where L_α acts on a function $h \in C_\natural^2(I_R)$ as follows:

$$(L_\alpha h)(x) = h''(x) + \frac{2\alpha+1}{x} h'(x) = \frac{1}{x^{2\alpha+1}} \frac{d}{dx}\big(x^{2\alpha+1} h'(x)\big). \tag{5}$$

Let J_ν be the Bessel function of the first kind of order ν,

$$\mathbb{I}_\nu(z) = \frac{J_\nu(z)}{z^\nu}, \quad z \in \mathbb{C},$$

$$\varphi_\lambda(x) = 2^\alpha \Gamma(\alpha+1) \mathbb{I}_\alpha(\lambda x), \quad x \in \mathbb{R}, \quad \lambda \in \mathbb{C}.$$

The function φ_λ is an eigenfunction of operators (5) and (1):

$$L_\alpha \varphi_\lambda = -\lambda^2 \varphi_\lambda, \quad T_x^\alpha \varphi_\lambda(y) = \varphi_\lambda(x)\varphi_\lambda(y) \tag{6}$$

(see [18, Chap. 6; 21, Sect. 2; 22, Sect. 2]). Using the Poisson integral representation for J_ν, it is easy to obtain the relation

$$\varphi_\lambda(x) = \int_0^x \cos(\lambda y) K(x, y) dy, \quad x > 0, \tag{7}$$

$$K(x, y) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \frac{(x^2 - y^2)^{\alpha - \frac{1}{2}}}{x^{2\alpha}}, \quad 0 \leqslant y < x,$$

and the following estimate

$$\left|\left(\frac{d}{dx}\right)^m \varphi_\lambda(x)\right| \leqslant \frac{\Gamma(\alpha+1)\Gamma(\frac{m+1}{2})}{\sqrt{\pi}\Gamma(\alpha+\frac{m}{2}+1)}|\lambda|^m e^{|x||\operatorname{Im}\lambda|}, \quad x\in\mathbb{R}, \quad \lambda\in\mathbb{C}, \quad m\in\mathbb{Z}_+. \tag{8}$$

The spherical transform (the Bessel transform) of a compactly supported function $f \in L^{1,\mathrm{loc}}_{\natural,\alpha}(\mathbb{R})$ is defined by

$$\widetilde{f}(\lambda) = \int_0^\infty f(x)\varphi_\lambda(x)x^{2\alpha+1}dx, \quad \lambda\in\mathbb{C}.$$

In this case, it follows from (3) and (6) that

$$\varphi_\lambda \star f = \widetilde{f}(\lambda)\varphi_\lambda, \quad \lambda\in\mathbb{C}. \tag{9}$$

Moreover, for compactly supported functions f, $g\in L^{1,\mathrm{loc}}_{\natural,\alpha}(\mathbb{R})$, we have an analogue of the Borel formula

$$\widetilde{f\star g}(\lambda) = \widetilde{f}(\lambda)\widetilde{g}(\lambda) \tag{10}$$

(see [21, Sect. 2; 22, Sect. 2]).

Let $C^m_{\natural,c}(\mathbb{R})$ be the space of even m times continuously differentiable functions on $\mathbb{R}$ with compact supports. For $f\in C_{\natural,c}(\mathbb{R})$, we set

$$r(f) = \inf\{r>0 : \operatorname{supp} f \subset \overline{I}_r\}.$$

Denote by $[x]$ and $\{x\}$ the integer and fractional parts of the number $x\in\mathbb{R}$ respectively.

Lemma 1. *Assume that $f\in C^m_{\natural,c}(\mathbb{R})$ for some $m\in\mathbb{Z}_+$, and let $D=\frac{d}{dx}$ be the differentiation operator. Then*

$$\widetilde{f}(\lambda) = \frac{(-1)^{[\frac{m}{2}]}}{\lambda^{2\left[\frac{m+1}{2}\right]}}\int_0^\infty \big(D^{2\{\frac{m}{2}\}}\varphi_\lambda\big)(x)\big(D^{2\{\frac{m}{2}\}}L_\alpha^{[\frac{m}{2}]}f\big)(x)x^{2\alpha+1}dx. \tag{11}$$

In particular,

$$|\widetilde{f}(\lambda)| \leqslant \frac{ce^{r(f)|\operatorname{Im}\lambda|}}{(1+|\lambda|)^m}, \quad \lambda\in\mathbb{C}, \tag{12}$$

where the constant $c>0$ does not depend on λ.

Proof. For $m=0$, relation (11) coincides with the definition of a spherical transform. Hence, by integrating in parts using (5), (6) and induction, we obtain

$$\widetilde{f}(\lambda) = \frac{(-1)^{\frac{m}{2}}}{\lambda^{2\left[\frac{m+1}{2}\right]}}\int_0^\infty \varphi_\lambda(x)\big(L_\alpha^{\frac{m}{2}}f\big)(x)x^{2\alpha+1}dx, \quad \text{if} \quad m\in 2\,\mathbb{Z}_+,$$

and

$$\widetilde{f}(\lambda) = \frac{(-1)^{[\frac{m}{2}]}}{\lambda^{m+1}}\int_0^\infty \varphi'_\lambda(x)\big(L_\alpha^{\frac{m-1}{2}}f\big)'(x)x^{2\alpha+1}dx, \quad \text{if} \quad m\in 2\,\mathbb{Z}_+ + 1.$$

These two equations are equivalent to the relation (11). The estimate (12) follows from (11) and (8). □

Lemma 2. *Assume that* $f \in C^m_{\natural,c}(\mathbb{R})$ *for some integer* $m > 2\alpha + 2$, *and let*

$$\gamma_\alpha = \frac{1}{\left(2^\alpha \Gamma(\alpha+1)\right)^2}.$$

Then

$$f(x) = \gamma_\alpha \int_0^\infty \widetilde{f}(\lambda)\varphi_\lambda(x)\lambda^{2\alpha+1}d\lambda, \quad x \in \mathbb{R}, \tag{13}$$

where the integral in (13) *converges uniformly on* $\mathbb{R}$.

Proof. Under the condition $f \in C^\infty_{\natural,c}(\mathbb{R})$, the required formula is proved in [25, Theorem 2.3]. In the general case, we put

$$g(x) = \gamma_\alpha \int_0^\infty \widetilde{f}(\lambda)\varphi_\lambda(x)\lambda^{2\alpha+1}d\lambda. \tag{14}$$

It can be seen from (12) and (8) that the integral in (14) converges uniformly on $\mathbb{R}$ and $g \in C_\natural(\mathbb{R})$. By virtue of (9), for any function $h \in C^\infty_{\natural,c}(\mathbb{R})$ we have

$$(g \star h)(x) = \gamma_\alpha \int_0^\infty \widetilde{f}(\lambda)\widetilde{h}(\lambda)\varphi_\lambda(x)\lambda^{2\alpha+1}d\lambda.$$

On the other hand, since $f \star h \in C^\infty_{\natural,c}(\mathbb{R})$, then

$$(f \star h)(x) = \gamma_\alpha \int_0^\infty \widetilde{f \star h}(\lambda)\varphi_\lambda(x)\lambda^{2\alpha+1}d\lambda = \gamma_\alpha \int_0^\infty \widetilde{f}(\lambda)\widetilde{h}(\lambda)\varphi_\lambda(x)\lambda^{2\alpha+1}d\lambda$$

(see [25, Theorem 2.3] and (10)). From the last two relations and the arbitrariness of the function h, we conclude that $f = g$. □

The following statement is the well-known Titchmarsh theorem of supports (see, for example, [14, Part 1, Sect. 3.2]).

Lemma 3. *Assume that* $f_1, f_2 \in L^1(0,1)$, *and let*

$$\int_0^t f_1(x)f_2(t-x)dx = 0$$

for almost all $t \in (0,1)$. *Then* $\operatorname{supp} f_1 \subset [a,1]$ *and* $\operatorname{supp} f_2 \subset [b,1]$ *for some* $a, b \in [0,1]$ *such that* $a + b \geqslant 1$.

Lemma 4. *Assume that* $lr < R \leqslant (l+1)r$ *for some* $l \in \mathbb{N}$, *and let* $m > 2\alpha+2$. *If* $f \in V^m_r(I_R)$ *and* $f = 0$ *on* I_{lr}, *then* $f = 0$ *on* I_R.

Proof. For any $\varepsilon \in (0, R - lr)$, we consider a function $\eta_\varepsilon \in C^\infty_\natural(\mathbb{R})$ such that $\eta_\varepsilon = 1$ on $I_{R-\varepsilon}$ and $\eta_\varepsilon = 0$ on $\mathbb{R} \setminus I_{R-\frac{\varepsilon}{2}}$. We set

$$F = \begin{cases} \eta_\varepsilon f & \text{on } I_R \\ 0 & \text{on } \mathbb{R} \setminus I_R. \end{cases}$$

Then $F \in C^m_{\natural,c}(\mathbb{R})$, and $F = 0$ on I_{lr}. In view of Lemmas 1 and 2,

$$\sup_{\lambda\in\mathbb{R}} \big|\widetilde{F}(\lambda)\big|(1+|\lambda|)^m < +\infty$$

and

$$F(x) = \gamma_\alpha \int\limits_0^\infty \widetilde{F}(\lambda)\varphi_\lambda(x)\lambda^{2\alpha+1}d\lambda, \quad x \in \mathbb{R}. \tag{15}$$

We define the function $h \in C_\natural(\mathbb{R})$ by relation

$$h(x) = \int\limits_0^\infty \widetilde{F}(\lambda)\lambda^{2\alpha+1}\cos(\lambda x)d\lambda. \tag{16}$$

Using (15), (7) and Fubini's theorem, we obtain

$$F(x) = \gamma_\alpha \int\limits_0^x h(y)K(x,y)dy, \quad x > 0. \tag{17}$$

Since $F = 0$ on I_{lr}, then

$$h = 0 \quad \text{on} \quad I_{lr} \tag{18}$$

(see Lemma 3). Next, it follows from the definition of the function F and the hypothesis that

$$F \star \chi_r = 0 \quad \text{on} \quad I_{R-r-\varepsilon}.$$

Taking (15), (9), and (7) into account, we can write the convolution $F \star \chi_r$ as

$$(F\star\chi_r)(x) = \gamma_\alpha \int\limits_0^x \left(\int\limits_0^\infty \widetilde{\chi}_r(\lambda)\widetilde{F}(\lambda)\lambda^{2\alpha+1}\cos(\lambda y)d\lambda\right)K(x,y)dy, \quad x > 0.$$

Therefore, Lemma 3 and the equality

$$\widetilde{\chi}_r(\lambda) = \int\limits_0^r \varphi_\lambda(x)x^{2\alpha+1}dx$$

imply the relation

$$\int\limits_0^r \left(\int\limits_0^\infty \widetilde{F}(\lambda)\lambda^{2\alpha+1}\varphi_\lambda(t)\cos(\lambda y)d\lambda\right)t^{2\alpha+1}dt = 0, \quad y \in I_{R-r-\varepsilon}. \tag{19}$$

The inner integral in (19) is transformed as follows (see (7) and (16)):

$$\int\limits_0^\infty \widetilde{F}(\lambda)\lambda^{2\alpha+1}\varphi_\lambda(t)\cos(\lambda y)d\lambda = \int\limits_0^\infty \widetilde{F}(\lambda)\lambda^{2\alpha+1}\cos(\lambda y)\left(\int\limits_0^t \cos(\lambda z)K(t,z)dz\right)d\lambda =$$

$$= \frac{1}{2}\int\limits_0^t \left(\int\limits_0^\infty \widetilde{F}(\lambda)\lambda^{2\alpha+1}\big(\cos(\lambda(y+z))+\cos(\lambda(y-z))\big)d\lambda\right)K(t,z)dz =$$

$$= \frac{1}{2}\int\limits_0^t \big(h(y+z)+h(y-z)\big)K(t,z)dz.$$

Now from (19) and (18) we have

$$\int_0^r \left(\int_0^t h(y+z)K(t,z)dz\right)t^{2\alpha+1}dt = 0, \quad 0 \leqslant y < R-r-\varepsilon.$$

After changing the integration order, we arrive at the relation

$$\int_y^{y+r} h(u)H(u-y)du = 0, \quad (l-1)r < y < R-r-\varepsilon,$$

where

$$H(\xi) = \int_\xi^r K(t,\xi)t^{2\alpha+1}dt, \quad 0 < \xi < r.$$

Hence and from (18) we find

$$\int_0^t h(v+lr)H(r-t+v)dv = 0, \quad 0 < t < R-lr-\varepsilon.$$

This relation, Lemma 3 and equality (18) show that $h = 0$ on $(0, R-\varepsilon)$. Therefore, $F = 0$ on $(0, R-\varepsilon)$ (see (17)). To complete the proof, it remains to use the evenness of the function f and the arbitrariness of $\varepsilon \in (0, R-lr)$. □

Corollary 1. *Let $0 < r < R \leqslant +\infty$, and assume that $f \in V_r^m(I_R)$ for some $m > 2\alpha+2$. If $f = 0$ on I_r, then $f = 0$ on I_R.*

Proof. For $r < R \leqslant 2r$, the statement is obtained from Lemma 4 for $l = 1$. Suppose that it is valid for $R \leqslant kr$ with some $k \geqslant 2$, and let $kr < R \leqslant (k+1)r$. By the hypothesis, $f \in V_r^m(I_{kr})$ and $f = 0$ on I_r. By the induction assumption, we conclude that $f = 0$ on I_{kr}. Then, owing to Lemma 4, $f = 0$ on I_R. □

3. Proof of Theorem 1

We put

$$f_1 = f,$$

$$f_{q+1}(x) = \int_0^{|x|} \frac{1}{t^{2\alpha+1}}\left(\int_0^t y^{2\alpha+1}f_q(y)dy\right)dt, \quad x \in I_R, \quad q \in \mathbb{N}.$$

It is clear that all functions f_q are equal to zero on I_r. In addition,

$$f_q \in C^{2q-3}(I_R) \quad \text{for} \quad q \geqslant 2, \tag{20}$$

and

$$L_\alpha(f_{q+1}) = f_q, \quad q \in \mathbb{N} \tag{21}$$

in the space of even distributions on I_R (see (5)). In particular,

$$f = L_\alpha^{q-1}(f_q), \quad q \in \mathbb{N}.$$

We claim that $f_q \in V_r(I_R)$ for any $q \geqslant 1$. For $q = 1$, this follows from the hypothesis of the theorem and the definition of the function f_1. Suppose that $f_k \in V_r(I_R)$ for some $k \in \mathbb{N}$. Then (see (4) and (21))

$$L_\alpha(f_{k+1} \star \chi_r) = (L_\alpha f_{k+1}) \star \chi_r = f_k \star \chi_r = 0 \quad \text{on} \quad I_{R-r}.$$

Hence, it can be seen that

$$f_{k+1} \star \chi_r = \text{const} \quad \text{on} \quad I_{R-r}.$$

Since

$$(f_{k+1} \star \chi_r)(0) = \int_0^r f_{k+1}(x) x^{2\alpha+1} dx = 0,$$

we have $f_{k+1} \star \chi_r = 0$ on I_{R-r}. So, $f_q \in V_r(I_R)$ and $f_q = 0$ on I_r for any $q \in \mathbb{N}$. Now, bearing (20) in mind and using Corollary 1, we conclude that $f_q = 0$ on I_R for $q > \alpha + \frac{5}{2}$. Therefore, $f = L_\alpha^{q-1}(f_q) = 0$ on I_R. Thus, Theorem 1 is proved. □

Conclusion

The proof of Theorem 1 shows that the following more general result can be obtained by completely similar reasoning.

Theorem 2. *Let g be a nonzero function with a compact support of class $L^{1,\mathrm{loc}}_{\natural,\alpha}(\mathbb{R})$, and let $R \in (r(g), +\infty]$. Assume that $f \in L^{1,\mathrm{loc}}_{\natural,\alpha}(I_R)$, $f \star g = 0$ on $I_{R-r(g)}$ and $f = 0$ on $I_{r(g)}$. Then $f = 0$ on I_R.*

Various analogs of Theorem 2 for the usual convolution on the real axis are due to Yu. I. Lyubich [6, Theorem 1], A. F. Leontiev [10, Chap. 5], V. V. Volchkov [3, Chap. 13] and D. A. Zaraisky [13].

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