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BANACH LATTICES OF CONTINUOUS SECTIONS[1]

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The aim of this note is to outline some application of ample continuous Banach bundles to the theory of Banach lattices.

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1. Introduction

The study of Banach lattices in terms of sections of continuous Banach bundles has been started by Giertz [1, 2]. Later Gutman create the theory of ample (or complete) continuous Banach bundles [3] and measurable Banach bundles admitting lifting [4]. A portion of the Gutman's theory was specified in the case of bundles of measurable Banach lattices by Ganiev [5] and Kusraev [6]. The aim of this short note is to outline some additional possibilities of applying ample Banach bundles to the theory of Banach lattices. Recall some definitions.

A *bundle of Banach lattices* over a set Q is a mapping $\mathscr{X}$ defined on Q and sending every point $q \in Q$ to a Banach lattice $\mathscr{X}(q) := (\mathscr{X}(q), \|\cdot\|_q)$. Each space $\mathscr{X}(q)$ of a bundle $\mathscr{X}$ is called its *stalk* over q. A mapping u defined on a nonempty subset $D \subset Q$ is called a *section* over D, if $u(q) \in \mathscr{X}(q)$ for every $q \in D$. A section over Q is called *global*. If Q is endowed with some topology we call sections over comeager subsets of Q *almost global*.

Let $S(Q, \mathscr{X})$ stands for the set of all global sections of $\mathscr{X}$, endowed with the structure of a vector lattice by letting $u \leqslant v \Longleftrightarrow u(q) \leqslant v(q)$ $(\forall q \in Q)$, and $(\alpha u+\beta v)(q) = \alpha u(q)+\beta v)(q)$ $(q \in Q)$, where $\alpha, \beta \in \mathbb{R}$ and $u, v \in S(Q, \mathscr{X})$. For each section $u \in S(Q, \mathscr{X})$ we define its point-wise norm by $|||u||| : q \mapsto \|u(q)\|_{\mathscr{X}(q)}$ $(q \in Q)$. A set of sections $\mathscr{U} \subset S(Q, \mathscr{X})$ is called stalk-wise dense in $\mathscr{X}$ if the set $\{u(q) : u \in \mathscr{U}\}$ is dense in $\mathscr{X}(q)$ for every $q \in Q$.

2. Continuous bundles of Banach lattices

Let Q be a topological space and $\mathscr{X}$ be a bundle of Banach lattices over Q. A set of global sections $\mathscr{C} \subset S(Q, \mathscr{X})$ is called a *continuity structure* on $\mathscr{X}$, if it satisfy the conditions:

(a) $\mathscr{C}$ is a vector lattice, i. e. $\alpha c_1 + \beta c_2 \in \mathscr{C}$, $|c| \in \mathscr{C}$ for all $\alpha, \beta \in \mathbb{R}$ and $c_1, c_2 \in \mathscr{C}$;

(b) the point-wise norm $|||c||| : Q \to \mathbb{R}$ is continuous for every $c \in \mathscr{C}$;

(c) $\mathscr{C}$ is stalk-wise dense in $\mathscr{X}$.

If $\mathscr{C}$ is a continuity structure on $\mathscr{X}$ then the pair $(\mathscr{X}, \mathscr{C})$ is called a *continuous bundle of Banach lattices over* Q. More details see in [3] and [7]. Below $(\mathscr{X}, \mathscr{C})$ stands for a continuous bundle of Banach lattices over Q. We say that a section $u \in S(D, \mathscr{X})$ over $D \subset Q$ is $\mathscr{C}$-continuous at the point $q \in D$ if the function $|||u - c|||$ is continuous at q for every $c \in \mathscr{C}$. A section $u \in S(D, \mathscr{X})$ is *$\mathscr{C}$-continuous* if it is $\mathscr{C}$-continuous at every $q \in D$.

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Lemma. *Let $(\mathscr{X},\mathscr{C})$ be a continuous bundle of Banach lattices over Q. The set of all $\mathscr{C}$-continuous sections over $D \subset Q$ is a vector lattice.*

$\lhd$ It is obvious that the set of all $\mathscr{C}$-continuous sections is a vector space. Ensure that if a section u is $\mathscr{C}$-continuous then so is $|u| : q \mapsto |u(q)|$ $(q \in D)$. It is sufficient to prove that the function $\||u| - c\| : Q \to \mathbb{R}$ is continuous at an arbitrary $q \in D$, for every $c \in \mathscr{C}$. Put $\lambda := \| |u|(q) - c(q)\|$. We have to prove that, given $q \in D$ and $\epsilon > 0$, one can choose a neighborhood U of q such that $\lambda - \epsilon < \| |u|(p) - c(p)\| < \lambda + \epsilon$ for every $p \in U$.

Select a section $v \in \mathscr{C}$ satisfying $\|u(q) - v(q)\| < \epsilon/2$. Observe that $\| |u|(q) - |v|(q)\| \leqslant \|u(q) - v(q)\| < \epsilon/2$. Taking into consideration the continuity of the function $\| |v| - c\|$ and the estimate $\| |v|(q) - c(q)\| \leqslant \| |u|(q) - c(q)\| + \| |u|(q) - |v|(q)\| < \lambda + \epsilon/2$ we can find a neighborhood U_1 of q with $\| |v|(p) - c(p)\| < \epsilon/2$ for all $p \in U_1$.

Similarly, the estimate $\| |v|(q) - c(q)\| \geqslant \||u|(q) - c(q)\| - \| |u|(q) - |v|(q)\| \geqslant \lambda - \epsilon/2$ implies that $\| |v|(p) - c(p)\| \geqslant \lambda - \epsilon/2$ $(p \in U_2)$ for some neighborhood U_2 of q. Now, for all $p \in U := U_1 \cap U_2$ we can easily deduce

$$\lambda - \epsilon = (\lambda - \epsilon/2) - \epsilon/2 < \| |v|(p) - c(p)\| - \| |u|(p) - |v|(p)\| \leqslant \| |u|(p) - c(p)\|,$$
$$\| |u|(p) - c(p)\| \leqslant \| |v|(p) - c(p)\| + \| |u|(p) - |v|(p)\| < (\lambda + \epsilon/2) + \epsilon/2 = \lambda + \epsilon. \rhd$$

3. Banach lattices of sections

Suppose that Q is a nonempty Stonean space ($\equiv$ extremally disconnected and compact Hausdorff space). Consider a continuous Banach bundle $\mathscr{X}$ over Q. If u is an almost global section of the bundle $\mathscr{X}$ then the function $q \mapsto \|u(q)\|_q$ is defined and continuous on a comeager set $\mathrm{dom}(u) \subset Q$. Consequently, there exists a unique function $|u| \in C_\infty(Q)$ such that $|u|(q) = \|u(q)\|_q$ $(q \in \mathrm{dom}(u))$.

In the set of almost global sections $\mathfrak{M}(Q, \mathscr{X})$ we can define an equivalence relation by letting $u \sim v$ if $u(q) = v(q)$ whenever $q \in \mathrm{dom}(u) \cap \mathrm{dom}(v)$. Then equivalent u and v we have $|u| = |v|$; therefore, we may define $|\tilde{u}| := |u|$, where $\tilde{u}$ is the coset of the almost global section u. Denote by $C_\infty(Q, \mathscr{X})$ the quotient space $\mathfrak{M}(Q, \mathscr{X})/\sim$.

In each coset $\tilde{u}$, there exists a unique section $\bar{u} \in \tilde{u}$ such that $\mathrm{dom}(v) \subset \mathrm{dom}(\bar{u})$ for all $v \in \tilde{u}$. The section $\bar{u}$ is called extended. The space $C_\infty(Q, \mathscr{X})$ can be represented also as the space of all extended almost global sections of the bundle $\mathscr{X}$, see [3]. The set $C_\infty(Q, \mathscr{X})$ can be naturally equipped with the structure of lattice-normed lattice. For instance, the element $\tilde{u}+\tilde{v}$ is defined as the coset of the almost global section $q \mapsto u(q)+v(q)$ $(q \in \mathrm{dom}(u)\cap\mathrm{dom}(v))$. If E is an order ideal in $C_\infty(Q)$ then we assign $E(\mathscr{X}) := \{u \in C_\infty(Q, \mathscr{X}) : |u| \in E\}$.

Recall that a *Banach–Kantorovich space over* a Dedekind complete vector lattice E is a vector space X with a decomposable norm $|\cdot| : X \to E_+$ which is norm complete with respect to order convergence in E. Decomposability means that, given $e_1, e_2 \in E_+$ and $x \in X$ with $|x| = e_1 + e_2$, there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2$ and $|x_k| = e_k$ $(k := 1, 2)$. If a Banach–Kantorovich space is in addition a vector lattice with monotone norm then it's called a *Banach–Kantorovich lattice.* A Banach–Kantorovich lattice X can be endowed with a scalar norm $x \mapsto \|\cdot\| := \||\cdot|\|_E$, whenever E is a Banach lattice. The following result see in [3, 7].

Theorem 1. *Let $\mathscr{X}$ be a continuous bundle of Banach lattices over a Stonean space Q. Then $C_\infty(Q, \mathscr{X})$ is a Banach–Kantorovich lattice over $C_\infty(Q)$. If E is an order ideal in $C_\infty(Q)$ then $\big(E(\mathscr{X}), |\cdot|\big)$ is a Banach–Kantorovich lattice over E. If, in addition, E is Banach lattice, then $\big(E(\mathscr{X}), \|\cdot\|\big)$ is a Banach lattice.*

$\lhd$ We need only to put together the 'Banach part', given in [3] and [7, Theorem 2.4.7], and the above Lemma. $\rhd$

Theorem 2. *Every Banach–Kantorovich lattice X over an order dense ideal $E \subset C_\infty(Q)$ is isometrically lattice isomorphic to $E(\mathscr{X})$ for some complete continuous bundle $\mathscr{X}$ of Banach lattices over Q. Moreover, such a bundle $\mathscr{X}$ is unique to within isometrically lattice isomorphism.*

$\lhd$ The 'Banach part' follows again from [3] (see also [7, 2.4.10]). The rest is easily deduced on using the above Lemma. $\rhd$

Let Q be the Stone space of the Boolean algebra $B(\Omega)$ and $\tau : \Omega \to Q$ is the canonical immersion of Ω into Q corresponding to a fixed lifting τ of $L^\infty(\Omega)$. Let $\mathscr{Y}$ be a complete continuous bundle of Banach lattices over Q and $\mathscr{X} = \mathscr{Y} \circ \tau$. If $\mathscr{C}$ is a continuous structure in $\mathscr{Y}$, then the set $\mathscr{C} \circ \tau$ is a measurability structure in $\mathscr{X}$. The composite $v \circ \tau$ is a measurable section of $\mathscr{X}$ for every $v \in C_\infty(Q, \mathscr{Y})$, see [2, 1.2.7, 1.4.9, 2.5.8]. Let $C(Q, \mathscr{X})$ stands for the set of all global continuous sections of $\mathscr{X}$. The following result may be considered as a bridge between continuous and measurable bundles of Banach lattices.

Theorem 3. *Let (Ω, Σ, μ) be a measurable space with the direct sum property. The mapping $v \mapsto (v \circ \tau)^\sim$ is isometric lattice isomorphism of Banach–Kantorovich lattices $C_\infty(Q, \mathscr{Y})$ and $L^0(\Omega, \mathscr{X})$, associated with the isomorphism $(e \mapsto (e \circ \tau)^\sim) : C_\infty(Q) \to L^0(\Omega)$. The image of $C(Q, \mathscr{Y})$ under this isomorphism is $L^\infty(\Omega, \mathscr{X})$.*

$\lhd$ The 'Banach part' can be found in [4] (see also [2, 2.5.9]). The remaining is obvious. $\rhd$

Remark 1. The theory of ample continuous bundles of Banach lattices is parallel to that of liftable Banach bundles presented in [6]. In particular, the results from [6, Theorems 2.9, 2.10, 3.3] have their counterparts for ample continuous bundles of Banach lattices.

4. Representation of injective Banach lattices

A real Banach lattice X is said to be *injective* if, for every Banach lattice Y, every closed vector sublattice $Y_0 \subset Y$, and every positive linear operator $T_0 : Y_0 \to X$ there exists a positive linear extension $T : Y \to X$ with $\|T_0\| = \|T\|$. This concept was introduced by Lotz [8]. Important contributions are due to Cartwright [9] and Haydon [10]. A new source of insight into the structure of injectives is a Boolean-valued approach, see [11, 12].

A band projection π in a Banach lattice X is said to be an *M-projection* if $\|x\| = \max\{\|\pi x\|, \|\pi^\perp x\|\}$ for all $x \in X$, where $\pi^\perp := I_X - \pi$. The collection of all M-projections forms a subalgebra $\mathbb{M}(X)$ of the Boolean algebra of all band projections $\mathbb{P}(X)$ in X. The closed f-subalgebra in the center $\mathscr{Z}(X)$ generated by $\mathbb{M}(X)$ is denoted by $\mathscr{Z}_m(X)$.

A positive operator $T : X \to F$ is said to have the *Levi property* if $T(X)^{\perp\perp} = F$ and $\sup x_\alpha$ exists in X for every increasing net $(x_\alpha) \subset X_+$, provided that the net (Tx_α) is order bounded in F. A *Maharam operator* is an order continuous order intervals preserving ($\equiv T([0, x]) = [0, Tx]$ for all $x \in X_+$) operator. An operator $T : X \to Y$ is called lattice $\mathbb{B}$-*isometry*, if it is a lattice isometry and $b \circ T = T \circ b$ for all $b \in \mathbb{B}$.

Theorem 4. *If Φ is a strictly positive Maharam operator with the Levi property taking values in a Dedekind complete AM-space Λ with unit and $|||x||| = \|\Phi(|x|)\|_\infty$ $(x \in L^1(\Phi))$, then $(L^1(\Phi), |||\cdot|||)$ is an injective Banach lattice with $\mathbb{M}(L^1(\Phi)) = \mathbb{P}(\Lambda)$. Conversely, any injective Banach lattice X is lattice $\mathbb{B}$-isometric to $(L^1(\Phi), |||\cdot|||)$ for some strictly positive Maharam operator Φ with the Levi property taking values in a Dedekind complete AM-space Λ with unit, where $\mathbb{B} = \mathbb{M}(L^1(\Phi)) = \mathbb{P}(\Lambda)$.*

◁ See [12]; details can be found in [11]. ▷

Theorem 5. *Every injective Banach lattice X with $\Lambda = \mathscr{Z}_m(X) = C(Q)$ and $\mathbb{B} := \mathbb{P}(\Lambda)$ is lattice $\mathbb{B}$-isometric to $\Lambda(\mathscr{X})$ for some complete continuous bundle $\mathscr{X}$ of Banach lattices over Q such that all stalks $\mathscr{X}(q)$ $(q \in Q)$ are AL-spaces. Moreover, such a bundle $\mathscr{X}$ is unique to within isometrically lattice isomorphism.*

◁ The proof consists of a combination of the representation Theorems 2 and 4. ▷

Remark 2. This result was proved essentially by Gierz [1, 2] and Haydon [10]. The above approach enables us to settle also the uniqueness problem.

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БАНАХОВЫ РЕШЕТКИ НЕПРЕРЫВНЫХ СЕЧЕНИЙ

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Заметка представляет собой набросок некоторых приложений просторных банаховых расслоений к теории банаховых решеток.

Ключевые слова: банахова решетка, непрерывное банахово расслоение, сечение, инъективная банахова решетка.