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ANALYSIS OF SINGLE SERVER FEEDBACK RETRIAL QUEUE WITH BERNOULLI WORKING VACATION AND STARTING FAILURE

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Abstract

The suggested queueing model describes a single-server feedback retrial queueing system with starting failure, Bernoulli working vacation and vacation interruptions. The server departs on a working vacation as soon as orbit is empty. During the working vacation period, the server provides a slower level of service. The supplementary variable method was utilized to determine the steady-state probability-generating functions for the system and its orbit. If there are consumers in the system at the end of each vacation, the server becomes idle and ready to serve new customers. The average busy time and the average busy cycle are presented as important system performance indicators. Additionally, the adaptive neuro-fuzzy interface system has compared the numerical results with the neuro-fuzzy results. Finally, particle swarm optimization (PSO) were utilized to obtain the best (optimal) cost for the system in this study. We have examined the convergence of these optimization strategies.

Keywords: Retrial queues, Feedback, Supplementary variable technique, Starting Failure and Working Vacation,ANFIS.

1. Introduction

In a queueing system (QS), queues involving continuous tries occur when a consumer comes and identify the server is occupied. The client is instructed to leave the service region and join a virtual area referred to as the 'orbit'. Subsequently, the customer within the orbit can make a service request after a period of time. In a vacation periods, the server halts its service entirely, becoming unavailable to the primary clients for a short duration, which is termed a "vacation." However, during the working vacation (WV) period, the server provides services to consumers, albeit at a reduced service rate. Also, the server's vacation may be ignored if customers arrive during the vacation period, and the server may resume operation in its regularly scheduled manner. It is known as the vacation interruption(VI) strategy. Major uses for this QS include delivering network services, online services, file transfer services, mail services and so on. A more realistic RQ with feedback happens in many real-world scenarios; for instance, in multiple-access telecommunications systems, where data returned as failures is forwarded again, it may be treated as a retrial queue with feedback.

1.1. Survey of Literature

In an $M/G/1$ retrial queue (RQ) with general retrial times, consumers who find the server busy join the orbit according to the first-come,first-served (FCFS) principle as studied by Gomez-Corral [1]. Such an instance occurs in certain communication protocols, in production lines at stores, etc. The RQ has been extensively studied by Falin and Templeton [2], Artalejo and Corral [3], Artalejo [4], etc. Many authors have investigated a single server retrial queue (SSRQ) with WVs and VIs,including Zhang and Hou [16], Gao and Liu [6], Gao et al. [5], Zhang and Liu [22], and Rajadurai et al. ([7], [8], [9]). Mokaddis et al.[10] explored the $M/G/1$ retrial queue with Bernoulli feedback,Starting Failure (SF) and a single vacation (SV). Clients in orbit connect to the server via FCFS discipline, and an arbitrary distribution is assumed for the retry time. The server goes on vacation when there are no clients on the system. If the server comes back from vacation and there are no consumers, it waits for the first client to arrive on the system from the outside.

Krishna Kumar et al.[11] researched a RQ with feedback and a server exposed to SF, as well as a general stochastic decomposition rule for $M/G/1$ vacation models. Rajadurai [12] investigated a single server preemptive priority RQ based on Bernoulli working vacation (BWV) and VIs. Rajadurai et al.[13] explored a SSRQ system with BWV and VI. Performance indicators and analytical illustrations are provided. Jain and Kumar [26] analyzed bulk arrival general service RQ subject to balking, feedback and vacation interruption under multiple WV policy. Pazhani Bala Murugan and Keerthana [27] investigated an $M/G/1$ feedback RQ with WV and a waiting server. Keerthiga and Indhira [28] examined SSRQ with two phases of service for retrial customers. Agarwal et al. [29] discussed detection of optimal WV service rate for retrial priority G-queue with immediate Bernoulli feedback. Rachita Sethi et al.[17] researched a threshold-based repair facility for machining systems with a WV approach. WV was established to allow repairmen to offer service at a reduced rate as opposed to entirely discontinuing operations. The idea of F-policy is used to govern its arrival in the system. The implementation of a threshold N-policy to start the repair reduces the system's cost. Performance measurements are computed utilizing the 4^{th}-order Runge-Kutta approach, and the numerical findings obtained are compared to the adaptive neuro fuzzy inference system (ANFIS).

Charu Bhargava and Madhu Jain [18] studied the "Modelling and Analysis of a Markovian multi server queue with an (e,d) SV procedure, server failures and repairs". Some stationary performance indicators are established after service completion using the matrix geometric technique. Additionally,the direct search method is utilised to estimate the best no. of idle, vacationing, and total no. of servers at the most affordable price. Also, the acquired numerical outcomes have been compared using a soft computing technique (SCT) based on an ANFIS. Radhika Agarwal et al.[19] analyzed the performance metrics that are used in improving service standards using the SVT and compared the analytical outcomes to the neuro fuzzy outcomes via the ANFIS (SCT). In addition, single and bi-objective minimization issues are explored with minimum attained via "PSO and a multi-objective GA" respectively.

In this research, we have extended the work of Rajadurai et al. [13] by including the ideas of feedback and SF. By using PSO, we have also performed a cost analysis of the model under consideration. Because the suggested solution improves repeatedly and the system gives us the best option that is feasible, this approach has gained a lot of reputation in recent years. When it comes to queueing analysis, this technique may be used to get productive outcomes, whether the goal is to save overall costs or maximize performance metrics. This framework aids in our analysis of various real-world queuing scenarios, allowing us to enhance the customer experience. To that extent, this article contributes. In areas with heavy traffic and congestion, this kind of project is highly pertinent and beneficial.

To the best of author's knowledge, there has been no previous research that has examined in this work. Therefore, to fill up this gap, in this article, we consider the feedback RQ with WV and VI subjected to server breakdown and repair. SVT has been used, and for some of the variables, a 3D graphical representation has also been provided. To attain optimal operating conditions, minimize expected costs, and maximize economic performance PSO a well-known meta-heuristic technique are applied. The aforementioned framework may be used in a wide variety of situations, including but not limited to: telephone switching, telecommunications, computer networks, online ticket booking centers, aviation traffic control, quality control procedures, and inspection testing of items. The purpose of this investigation is to estimate the queue length and orbit size dist., which will be implemented to calculate the system's performance metrics.

The following is an overview of our article: Section 2 provides a detailed discussion of the queueing paradigm. Section 3 specifically determines the system's steady state (SS) behavior and the queue length's PGF at a random epoch. Section 4 includes various substantial indicators of system behavior. Section 5 discussed particular cases. Sections 6 and 7 provide numerical outcomes and cost optimization. Finally, Section 8 provides a conclusion and overview of the study.

2. Model description

A comprehensive explanation of this framework is given below:

The arrival process: New customers join the system from the outside, according to a Poisson process (PP), at a rate of μ.

The retrial process: According to FCFS discipline, when a customer visits while the server is occupied or unavailable, the consumer departs the service area and joins a group of blocked clients known as "orbit." The server appears to be accessible to the clients at the front of the orbit queue. Every customer's successive inter-retrial duration's are determined by an arbitrary probability distribution function (PDF) B(x), with an associated density function (df) b(x) and the "Laplace-Stieltjes transform" (LST) $\beta^*(\theta)$.

The service process: When a new or repeated consumers enters at the server while it's idle, the server promptly begins its regular service for the incoming customers. The service time follows a general dist., and PDF $C(x)$, a df $c(x)$, a LST $\alpha^*(\theta)$ & the 1^{st} and 2^{nd} moments are C_1 and C_2.

The Bernoulli working vacation process: The server goes on a WV whenever the orbit is empty, and the duration of this vacation follows an exponential dist., with a specified parameter γ. If a consumer visits during vacation time, the server will continue to operate at a slower service rate. The WV time is a slower-paced operating period. In the event that any clients in the orbit reach the instant of service completion during the vacation time, the server will end the vacation and return to its normally busy state, which is known as VI. On the other hand, if there are no clients in the system at the completion of the vacation, the server rejoins the system and waits to serve a new client with prob.r_1 (SWV) or makes for another WV with prob. $r_2 = 1 - r_1$ (MWV). When a vacation is over and there are still consumers in its orbit, the server resumes usual operation. During the WV period, the service time is determined by a general random variable H_v with a dist., function $H_v(t)$, LST $H_v^*(\varphi)$ & the 1^{st} and 2^{nd} moments are h_1 and h_2, respectively.

Feedback Procedure: After getting their normal services, dissatisfied customers have two options: they may either exit the system with probability $\bar{\omega} = (1 - \omega)$ or they can return to the orbit as unsatisfied clients and get a service again with probability ω .

Starting Failure: The customer will almost definitely start receiving service right away if the server is successfully activated. If the server is unable to start the service, the consumer exits the service area, enters the orbit, and repeats the request for the service after some time. The server is instantly repaired if a failure occurs. SF happens with prob. $\bar{\lambda}$ and successful service begins with prob.λ.

Repair Process: If the server fails to start, the repair process begins immediately. During the repair process, the server refuses to serve external or repeat consumers. Repair times have a distribution function I(x) and a corresponding density function i(x), and the first two moments are I_1 and I_2, respectively.

The system's stochastic processes are considered to be independent of each other.

2.1. Practical justification of the recommended paradigm

The suggested scenario has beneficial applications in a telecommunications. For example, we investigate a communication system designed for making reservations at restaurants. Let us consider a scenario in which a restaurant uses a phone system to accept reservations and provide a range of other services. client can use this system to reserve a table for themselves. The supervisor who answers all calls is in charge of this phone system. The consumer is able to pick up the phone (leave the system) or inquire regarding event reservations, purchase tickets for an upcoming musical performance, etc. after reserving a table. A caller must reaffirm their reservations if there is a possibility of a misinterpretation stemming from an unclear network or other related difficulties (feedback). When the manager is occupied overseeing other areas of the restaurant, he is unable to answer calls (vacation mode). In these circumstances, the junior manager often serves, albeit somewhat more slowly (WV).In this stage, the supervisor returns right away (i.e., a vacation interruption happens) if there are any calls in the system after the phone call is over (at service completion). However, the supervisor continues to take care of other restaurant-related matters if no calls come in after completing his secondary work (vacation mode). It is likely that when a consumer calls, the line is busy and that the client will call back after some period of time (retrial). It is possible that during a phone conversation, a bad signal, inadequate network coverage, or a virus attack (SF) might occur, causing the client to lose service. Once the communication system's signal is repaired, it functions flawlessly.

3. Scrutiny of the steady state probabilities

In steady state (SS), we presume that $\mathcal{B}(0) = 0$, $\mathcal{B}(\infty) = 1$, $\mathcal{C}(0) = 0$, $\mathcal{C}(\infty) = 1$, and $\mathcal{H}_v(0) = 0$, $\mathcal{H}_v(\infty) = 1$, $\mathcal{I}(0) = 0$,$\mathcal{I}(\infty) = 1$, are continuous at $\tilde{\varphi} = 0$. So that the function $\eta(\tilde{\varphi})$, $\zeta(\tilde{\varphi})$, $\kappa(\tilde{\varphi})$, and $v(\tilde{\varphi})$, are the hazard rates of the conditions (retrial, normal service,vacation and repair) are

$$\eta(\tilde{\varphi})d\tilde{\varphi} = \frac{d\mathcal{B}(\tilde{\varphi})}{1-\mathcal{B}(\tilde{\varphi})}$$
$$\zeta(\tilde{\varphi})d\tilde{\varphi} = \frac{d\mathcal{C}(\tilde{\varphi})}{1-\mathcal{C}(\tilde{\varphi})}$$
$$\kappa(\tilde{\varphi})d\tilde{\varphi} = \frac{d\mathcal{H}_v(\tilde{\varphi})}{1-\mathcal{H}_v(\tilde{\varphi})}$$
$$v(\tilde{\varphi})d\tilde{\varphi} = \frac{d\mathcal{I}(\tilde{\varphi})}{1-\mathcal{I}(\tilde{\varphi})}$$

$$L(\xi) = \begin{cases} 0, & \text{if the server is free} \\ 1, & \text{if the server is active period} \\ 2, & \text{if the server is operative mode on WV period} \\ 3, & \text{if the server is on repair} \end{cases}$$

Thus, the state of the system $\mathcal{B}^0(\xi)$,$\mathcal{C}^0(\xi)$, $\mathcal{H}_v^0(\xi)$, and $\mathcal{I}^0(\xi)$ are required to construct a bivariate Markov process $\{N(\xi); \xi \geq 0\}$, where $L(\xi)$ belongs to the server stage $(0, 1, 2, 3)$ based on if the server is idle, typical operative period,slow service and repair time.

3.1. Ergodicity Condition

Let $\{\xi_\sigma; \sigma = 1, 2, ...\}$ represent a series of epochs in which either a service time is reduced or completed. $U_\sigma = \{L(\xi_\sigma+), X(\xi_\sigma+)\}$ is a random vector sequence. The embedded Markov chain generated by the RQ system. Its state space is S={0,1,2,3} x N.

3.2. Theorem

The embedded Markov chain $\{U_\sigma; \sigma \epsilon N\}$ is ergodic iff $\rho < \bar{\mathcal{B}}(\mu)$ for our system to be stable, where $\rho = \lambda\mu\mathcal{C}_1 + \bar{\lambda}(1 + \mu\mathcal{I}_1) + \lambda\omega$.

3.3. System of governing equations

For the procedure $\{N(\xi), \xi \geq 0\}$, we specify the prob., $\phi_0(\xi) = P\{L(\xi) = 0, X(\xi) = 0\}$ and $\chi_0(\xi) = P\{L(\xi) = 1, X(\xi) = 0\}$ the probability densities,
$\chi_\sigma(\tilde{\varphi}, \xi)d\tilde{\varphi} = P\{L(\xi) = 1, X(\xi) = \sigma, \tilde{\varphi} \leq \mathcal{B}^0(\xi) < \tilde{\varphi} + d\tilde{\varphi}\}$,
for $\xi \geq 0$, $\tilde{\varphi} \geq 0$ and $\sigma \geq 1$.
$\Psi_\sigma(\tilde{\varphi}, \xi)d\tilde{\varphi} = P\{L(\xi) = 2, X(\xi) = \sigma, \tilde{\varphi} \leq \mathcal{C}^0(\xi) < \tilde{\varphi} + d\tilde{\varphi}\}$,
for $\xi \geq 0$, $\tilde{\varphi} \geq 0$, $\sigma \geq 0$.
$\Lambda_{v,\sigma}(\tilde{\varphi}, \xi)d\tilde{\varphi} = P\{L(\xi) = 3, X(\xi) = \sigma, \tilde{\varphi} \leq \mathcal{H}_v^0(\xi) < \tilde{\varphi} + d\tilde{\varphi}\}$,
for $\xi \geq 0$, $\tilde{\varphi} \geq 0$ and $\sigma \geq 0$.
$\Pi_\sigma(\tilde{\varphi}, \xi)d\tilde{\varphi} = P\{L(\xi) = 4, X(\xi) = \sigma, \tilde{\varphi} \leq \mathcal{I}^0(\xi) < \tilde{\varphi} + d\tilde{\varphi}\}$,
for $\xi \geq 0$, $\tilde{\varphi} \geq 0$, $\sigma \geq 1$.

In subsequent parts, the following probabilities are applied:

1. The prob., of the server being idle and on WV at time ξ is denoted by $\phi_0(\xi)$.
2. The prob., of the server being idle and on typical active period at time ξ is denoted by $\chi_0(\xi)$.
3. If there are accurately σ clients in the orbit at time ξ and the elapsed retrial time of the test clients undergoing retrial is between $\tilde{\varphi}$ and $\tilde{\varphi} + d\tilde{\varphi}$, then the prob., that this is the case is $\chi_\sigma(\tilde{\varphi}, \xi)$.
4. When there are σ consumers in the orbit, the prob., of the test customer's elapsed regular service time ranging between $\tilde{\varphi}$ and $\tilde{\varphi} + d\tilde{\varphi}$ is $\Psi_\sigma(\tilde{\varphi}, \xi)$.
5. $\Lambda_{v,\sigma}(\tilde{\varphi}, \xi)d\tilde{\varphi}$ and $\Pi_\sigma(\tilde{\varphi}, \xi)d\tilde{\varphi}$ is the prob., that there are precisely σ patrons in the orbit, with the elapsed (reduced service time and repair time) of the test patron being between $\tilde{\varphi}$ and $\tilde{\varphi} + d\tilde{\varphi}$ at time ξ.

Suppose that the sequel fulfills the stability condition, thus we can provide $\chi_0 = lim_{\xi\to\infty}\chi_0(\xi)$ and limiting densities are
$\chi_\sigma(\tilde{\varphi}) = lim_{\xi\to\infty}\chi_\sigma(\tilde{\varphi}, \xi)$ for $\tilde{\varphi} \geq 0$ and $\sigma \geq 1$.
$\Psi_\sigma(\tilde{\varphi}) = lim_{\xi\to\infty}\Psi_\sigma(\tilde{\varphi}, \xi)$ for $\tilde{\varphi} \geq 0$ and $\sigma \geq 0$.
$\Lambda_{v,\sigma}(\tilde{\varphi}) = lim_{\xi\to\infty}\Lambda_{v,\sigma}(\tilde{\varphi}, \xi)$ for $\tilde{\varphi} \geq 0$ and $\sigma \geq 0$.
$\Pi_\sigma(\tilde{\varphi}) = lim_{\xi\to\infty}\Pi_\sigma(\tilde{\varphi}, \xi)$ for $\tilde{\varphi} \geq 0$ and $\sigma \geq 1$.

Applying the SVT, we create the following system of equations.

$$\mu\chi_0 = \gamma r_1\phi_0 \tag{1}$$

$$(\mu+\gamma)\phi_0 = \gamma r_2\phi_0 \int_0^\infty \Lambda_{v,0}(\tilde{\varphi})\kappa(\tilde{\varphi})d\tilde{\varphi} + \int_0^\infty \Psi_\sigma(\tilde{\varphi})\zeta(\tilde{\varphi})d\tilde{\varphi} \tag{2}$$

$$\frac{d}{d\tilde{\varphi}}\chi_\sigma(\tilde{\varphi}) + (\mu+\eta(\tilde{\varphi}))\chi_\sigma(\tilde{\varphi}) = 0, \sigma \geq 1 \tag{3}$$

$$\frac{d}{d\tilde{\varphi}}\Psi_0(\tilde{\varphi}) + (\mu+\zeta(\tilde{\varphi}))\Psi_0(\tilde{\varphi}) = 0, \sigma = 0. \tag{4}$$

$$\frac{d}{d\tilde{\varphi}}\Psi_\sigma(\tilde{\varphi}) + (\mu+\zeta(\tilde{\varphi}))\Psi_\sigma(\tilde{\varphi}) = \mu\Psi_{\sigma-1}(\tilde{\varphi}), \sigma \geq 1 \tag{5}$$

$$\frac{d}{d\tilde{\varphi}}\Lambda_{0,v}(\tilde{\varphi}) + (\mu+\gamma+\kappa(\tilde{\varphi}))\Lambda_{0,v}(\tilde{\varphi}) = 0, \sigma = 0. \tag{6}$$

$$\frac{d}{d\tilde{\varphi}}\Lambda_{\sigma,v}(\tilde{\varphi}) + (\mu+\gamma+\kappa(\tilde{\varphi}))\Lambda_{\sigma,v}(\tilde{\varphi}) = \mu\Lambda_{v,\sigma-1}(\tilde{\varphi}), \sigma \geq 1. \tag{7}$$

$$\frac{d}{d\tilde{\varphi}}\Pi_0(\tilde{\varphi}) + (\mu+v(\tilde{\varphi}))\Pi_0(\tilde{\varphi}) = 0, \sigma = 0. \tag{8}$$

$$\frac{d}{d\tilde{\varphi}}\Pi_\sigma(\tilde{\varphi}) + (\mu+v(\tilde{\varphi}))\Pi_\sigma(\tilde{\varphi}) = \mu\Pi_{\sigma-1}(\tilde{\varphi}), \sigma \geq 1. \tag{9}$$

At $\tilde{\varphi} = 0$ the steady state boundary conditions are as follows:

$$\begin{aligned}\chi_\sigma(0) = \bar{\omega}\int_0^\infty \Psi_\sigma(\tilde{\varphi})\zeta(\tilde{\varphi})d\tilde{\varphi} + \omega\int_0^\infty \Psi_{\sigma-1}(\tilde{\varphi})\zeta(\tilde{\varphi})d\tilde{\varphi} + \bar{\omega}\int_0^\infty \Lambda_{v,\sigma}(\tilde{\varphi})\kappa(\tilde{\varphi})d\tilde{\varphi} \\ + \omega\int_0^\infty \Lambda_{v,\sigma-1}(\tilde{\varphi})\kappa(\tilde{\varphi})d\tilde{\varphi} + \int_0^\infty \Pi_\sigma(\tilde{\varphi})v(\tilde{\varphi})d\tilde{\varphi}\end{aligned} \tag{10}$$

$$\Psi_0(0) = \lambda\int_0^\infty \chi_1(\tilde{\varphi})\eta(\tilde{\varphi})d\tilde{\varphi} + \lambda\bar{\mu}\chi_0 + \gamma\int_0^\infty \Lambda_{0,v}(\tilde{\varphi})d\tilde{\varphi}, \sigma = 0 \tag{11}$$

$$\Psi_\sigma(0) = \lambda\int_0^\infty \chi_{\sigma+1}(\tilde{\varphi})\eta(\tilde{\varphi})d\tilde{\varphi} + \lambda\mu\int_0^\infty \Psi_\sigma(\tilde{\varphi})d\tilde{\varphi} + \gamma\int_0^\infty \Lambda_{\sigma,v}(\tilde{\varphi})d\tilde{\varphi}, \sigma \geq 1 \tag{12}$$

$$\Lambda_{v,\sigma}(0) = \begin{cases} \mu\phi_0, & \sigma = 0 \\ 0, & \sigma \geq 1 \end{cases} \tag{13}$$

$$\Pi_1(0) = \bar{\lambda}\int_0^\infty \chi_1(\tilde{\varphi})\eta(\tilde{\varphi})d\tilde{\varphi} + \bar{\lambda}\mu\chi_0 \tag{14}$$

$$\Pi_\sigma(0) = \bar{\lambda}\int_0^\infty \chi_\sigma(\tilde{\varphi})\eta(\tilde{\varphi})d\tilde{\varphi} + \bar{\lambda}\mu\int_0^\infty \chi_{n-1}(\tilde{\varphi})d\tilde{\varphi}, \sigma \geq 2 \tag{15}$$

The normalizing condition is

$$\begin{aligned}\chi_0 + \phi_0 + \sum_{\sigma=1}^\infty\int_0^\infty \chi_\sigma(\tilde{\varphi})d\tilde{\varphi} + \sum_{\sigma=0}^\infty\int_0^\infty \Psi_\sigma(\tilde{\varphi})d\tilde{\varphi} + \sum_{\sigma=0}^\infty\int_0^\infty \Lambda_{\sigma,v}(\tilde{\varphi})d\tilde{\varphi} \\ + \sum_{\sigma=1}^\infty\int_0^\infty \Pi_\sigma(\tilde{\varphi})d\tilde{\varphi} = 1\end{aligned} \tag{16}$$

3.4. The steady state solution

The PGF is used to compute the steady state solution for the RQ model. To solve the aforementioned equations, the generating functions for $|\vartheta| < 1$ are described as below:

$$\chi(\tilde{\varphi},\vartheta) = \sum_{\sigma=1}^{\infty} \chi_\sigma(\tilde{\varphi})\vartheta^\sigma; \chi(0,\vartheta) = \sum_{\sigma=1}^{\infty} \chi_\sigma(0)\vartheta^\sigma;$$

$$\Psi(\tilde{\varphi},\vartheta) = \sum_{\sigma=0}^{\infty} \Psi_\sigma(\tilde{\varphi})\vartheta^\sigma; \Psi(0,\vartheta) = \sum_{n=0}^{\infty} \Psi_0(0)\vartheta^\sigma; i = 1,2$$

$$\Lambda_v(\tilde{\varphi},\vartheta) = \sum_{\sigma=0}^{\infty} \Lambda_{v,\sigma}(\tilde{\varphi})\vartheta^\sigma; \Lambda_v(0,\vartheta) = \sum_{\sigma=0}^{\infty} \Lambda_{v,\sigma}(0)\vartheta^\sigma;$$

$$\Pi(\tilde{\varphi},\vartheta) = \sum_{\sigma=1}^{\infty} \Pi_\sigma(\tilde{\varphi})\vartheta^\sigma; \Pi(0,\vartheta) = \sum_{\sigma=1}^{\infty} \Pi_\sigma(0)\vartheta^\sigma$$

Next multiply the SS eqn. and SS boundary conditions from (3) to (15) by ϑ^σ and adding over σ, $(\sigma = 0, 1, 2, ...)$

$$\frac{\partial}{\partial\tilde{\varphi}}\chi(\tilde{\varphi},\vartheta) + [\mu + \eta(\tilde{\varphi})]\chi(\tilde{\varphi},\vartheta) = 0 \tag{17}$$

$$\frac{\partial}{\partial\tilde{\varphi}}\Psi(\tilde{\varphi},\vartheta) + [\mu(1-\vartheta) + \zeta(\tilde{\varphi})]\Psi(\tilde{\varphi},\vartheta) = 0 \tag{18}$$

$$\frac{\partial}{\partial\tilde{\varphi}}\Lambda_v(\tilde{\varphi},\vartheta) + [\gamma + \mu(1-\vartheta) + \kappa(\tilde{\varphi})]\Lambda_v(\tilde{\varphi},\vartheta) = 0 \tag{19}$$

$$\frac{\partial}{\partial\tilde{\varphi}}\Pi(\tilde{\varphi},\vartheta) + [\mu(1-\vartheta) + v(\tilde{\varphi})]\Pi(\tilde{\varphi},\vartheta) = 0 \tag{20}$$

Solving the partial differential eqns. (17) to (20), we obtain

$$\chi(\tilde{\varphi},\vartheta) = \chi(0,\vartheta)[1-\mathcal{B}(\tilde{\varphi})]e^{-\mu\tilde{\varphi}} \tag{21}$$

$$\Psi(\tilde{\varphi},\vartheta) = \Psi(0,\vartheta)[1-\mathcal{C}(\tilde{\varphi})]e^{-\mathcal{F}(\vartheta)\tilde{\varphi}} \tag{22}$$

$$\Lambda_v(\tilde{\varphi},\vartheta) = \Lambda_v(0,\vartheta)[1-\mathcal{H}_v(\tilde{\varphi})]e^{-\mathcal{F}_v(\vartheta)\tilde{\varphi}} \tag{23}$$

$$\Pi(\tilde{\varphi},\vartheta) = \Pi(0,\vartheta)[1-\mathcal{I}(\tilde{\varphi})]e^{-\mathcal{F}(\vartheta)\tilde{\varphi}} \tag{24}$$

where $\mathcal{F}(\vartheta) = \mu(1-\vartheta)$, $\mathcal{F}_v(\vartheta) = \gamma + \mu(1-\vartheta)$

Multiplying equation (10) and (12,13,15) by appropriate powers of ϑ, adding over n with few mathematical manipulations, we obtain

$$\begin{aligned}\chi(0,\vartheta) = (\bar{\omega} + \omega\vartheta)\int_0^\infty \Psi(\tilde{\varphi},\vartheta)\zeta(\tilde{\varphi})d\tilde{\varphi} + (\bar{\omega} + \omega\vartheta)\int_0^\infty \Lambda_v(\tilde{\varphi},\vartheta)\kappa(\tilde{\varphi})d\tilde{\varphi} \\ + \int_0^\infty \Pi(\tilde{\varphi},\vartheta)v(\tilde{\varphi})d\tilde{\varphi} - (\mu + \gamma r_1)\phi_0\end{aligned} \tag{25}$$

$$\Psi(0,\vartheta) = \frac{\lambda}{\vartheta}\int_0^\infty \chi(\tilde{\varphi},\vartheta)\eta(\tilde{\varphi})d\tilde{\varphi} + \lambda\mu\int_0^\infty \chi(\tilde{\varphi},\vartheta)d\tilde{\varphi} + \gamma\int_0^\infty \Lambda_v(\tilde{\varphi},\vartheta)d\tilde{\varphi} + \lambda\mu\chi_0 \tag{26}$$

$$\Lambda_v(0,\vartheta) = \mu\phi_0 \tag{27}$$

$$\Pi(0,\vartheta) = \bar{\lambda}\vartheta\mu\int_0^\infty \chi(\tilde{\varphi},\vartheta)d\tilde{\varphi} + \bar{\lambda}\int_0^\infty \chi(\tilde{\varphi},\vartheta)\eta(\tilde{\varphi})d\tilde{\varphi} + \vartheta\mu\bar{\lambda}\chi_0 \tag{28}$$

Using eqn (21,23 and 27) in eqn (26)

$$\Psi(0,\vartheta) = \lambda\chi(0,\vartheta)\left[\frac{\vartheta + (1-\vartheta)\bar{\mathcal{B}}(\mu)}{\vartheta}\right] + \mu\phi_0\mathcal{V}(\vartheta) + \lambda\gamma r_1\phi_0 \tag{29}$$

Similarly using equation (21) in (28)

$$\Pi(0,\vartheta) = \vartheta\gamma r_1\bar{\lambda}\chi_0 + \bar{\lambda}\chi(0,\vartheta)[\vartheta + (1-\vartheta)\bar{\mathcal{B}}(\mu)] \tag{30}$$

Substituting equations (22),(23) and (24) in (25), we obtain

$$\chi(0,\vartheta) = (\bar{\omega}+\omega\vartheta)\Psi(0,\vartheta)\bar{\mathcal{C}}(\mathcal{F}(\vartheta)) + (\bar{\omega}+\omega\vartheta)\Lambda_v(0,\vartheta)\bar{\mathcal{H}}_v(\mathcal{F}_v(\vartheta)) + \Pi(0,\vartheta)\bar{\mathcal{I}}(\mathcal{F}(\vartheta)) - \mu\phi_0 - \gamma r_1\phi_0 \tag{31}$$

Using equations (27),(29) and (30) in equation (31)

$$\chi(0,\vartheta) = \vartheta\left\{\frac{\begin{array}{r}(\bar{\omega}+\omega\vartheta)\mu\phi_0[\bar{\mathcal{H}}_v(\mathcal{F}_v(\vartheta)) + \mathcal{V}(\vartheta)\bar{\mathcal{C}}(\mathcal{F}(\vartheta))] + (\bar{\omega}+\omega\vartheta)\lambda\gamma r_1\phi_0\bar{\mathcal{C}}(\mathcal{F}(\vartheta))\\ + \vartheta\bar{\lambda}\gamma r_1\phi_0\bar{\mathcal{I}}(\mathcal{F}(\vartheta)) - \mu\phi_0 - \gamma r_1\phi_0\end{array}}{\vartheta - [(\bar{\omega}+\omega\vartheta)\lambda\bar{\mathcal{C}}(\mathcal{F}(\vartheta)) + \vartheta\bar{\lambda}\bar{\mathcal{I}}(\mathcal{F}(\vartheta))][\vartheta + (1-\vartheta)\bar{\mathcal{B}}(\mu)]}\right\} \tag{32}$$

substituting equation (32) in (29) and (30),we obtain

$$\Psi(0,\vartheta) = \left\{\frac{\begin{array}{r}\lambda[(\bar{\omega}+\omega\vartheta)\mu\phi_0[\bar{\mathcal{H}}_v(\mathcal{F}_v(\vartheta)) + \mathcal{V}(\vartheta)\bar{\mathcal{C}}(\mathcal{F}(\vartheta))] + (\bar{\omega}+\omega\vartheta)\lambda\gamma r_1\phi_0\bar{\mathcal{C}}(\mathcal{F}(\vartheta))\\ + \vartheta\bar{\lambda}\gamma r_1\phi_0\bar{\mathcal{I}}(\mathcal{F}(\vartheta)) - \mu\phi_0 - \gamma r_1\phi_0][\vartheta + (1-\vartheta)\bar{\mathcal{B}}(\mu)] + \mu\phi_0\mathcal{V}(\vartheta) + \lambda\gamma r_1\phi_0\end{array}}{\vartheta - [(\bar{\omega}+\omega\vartheta)\lambda\bar{\mathcal{C}}(\mathcal{F}(\vartheta)) + \vartheta\bar{\lambda}\bar{\mathcal{I}}(\mathcal{F}(\vartheta))][\vartheta + (1-\vartheta)\bar{\mathcal{B}}(\mu)]}\right\} \tag{33}$$

$$\Pi(0,\vartheta) = \left\{\frac{\begin{array}{r}\bar{\lambda}[(\bar{\omega}+\omega\vartheta)\mu\phi_0[\bar{\mathcal{H}}_v(\mathcal{F}_v(\vartheta)) + \mathcal{V}(\vartheta)\bar{\mathcal{C}}(\mathcal{F}(\vartheta))] + (\bar{\omega}+\omega\vartheta)\lambda\gamma r_1\phi_0\bar{\mathcal{C}}(\mathcal{F}(\vartheta))\\ + \vartheta\bar{\lambda}\gamma r_1\phi_0\bar{\mathcal{I}}(\mathcal{F}(\vartheta)) - \mu\phi_0 - \gamma r_1\phi_0] + \vartheta\bar{\lambda}\gamma r_1\phi_0\end{array}}{\vartheta - [(\bar{\omega}+\omega\vartheta)\lambda\bar{\mathcal{C}}(\mathcal{F}(\vartheta)) + \vartheta\bar{\lambda}\bar{\mathcal{I}}(\mathcal{F}(\vartheta))][\vartheta + (1-\vartheta)\bar{\mathcal{B}}(\mu)]}\right\} \tag{34}$$

Substituting equations (27) and (32) to (34) in (21) to (24)

$$\chi(\tilde{\varphi},\vartheta) = \vartheta\left\{\frac{\begin{array}{r}(\bar{\omega}+\omega\vartheta)\mu\phi_0[\bar{\mathcal{H}}_v(\mathcal{F}_v(\vartheta)) + \mathcal{V}(\vartheta)\bar{\mathcal{C}}(\mathcal{F}(\vartheta))] + (\bar{\omega}+\omega\vartheta)\lambda\gamma r_1\phi_0\bar{\mathcal{C}}(\mathcal{F}(\vartheta))\\ + \vartheta\bar{\lambda}\gamma r_1\phi_0\bar{\mathcal{I}}(\mathcal{F}(\vartheta)) - \mu\phi_0 - \gamma r_1\phi_0\end{array}}{\vartheta - [(\bar{\omega}+\omega\vartheta)\lambda\bar{\mathcal{C}}(\mathcal{F}(\vartheta)) + \vartheta\bar{\lambda}\bar{\mathcal{I}}(\mathcal{F}(\vartheta))][\vartheta + (1-\vartheta)\bar{\mathcal{B}}(\mu)]}\right\} \times [1-\mathcal{B}(\tilde{\varphi})]e^{-\mu\tilde{\varphi}} \tag{35}$$

$$\Psi(\tilde{\varphi},\vartheta) = \left\{\frac{\begin{array}{r}\lambda[(\bar{\omega}+\omega\vartheta)\mu\phi_0[\bar{\mathcal{H}}_v(\mathcal{F}_v(\vartheta)) + \mathcal{V}(\vartheta)\bar{\mathcal{C}}(\mathcal{F}(\vartheta))] + (\bar{\omega}+\omega\vartheta)\lambda\gamma r_1\phi_0\bar{\mathcal{C}}(\mathcal{F}(\vartheta))\\ + \vartheta\bar{\lambda}\gamma r_1\phi_0\bar{\mathcal{I}}(\mathcal{F}(\vartheta)) - \mu\phi_0 - \gamma r_1\phi_0][\vartheta + (1-\vartheta)\bar{\mathcal{B}}(\mu)] + \mu\phi_0\mathcal{V}(\vartheta) + \lambda\gamma r_1\phi_0\end{array}}{\vartheta - [(\bar{\omega}+\omega\vartheta)\lambda\bar{\mathcal{C}}(\mathcal{F}(\vartheta)) + \vartheta\bar{\lambda}\bar{\mathcal{I}}(\mathcal{F}(\vartheta))][\vartheta + (1-\vartheta)\bar{\mathcal{B}}(\mu)]}\right\} \times [1-\mathcal{C}(\tilde{\varphi})]e^{-\mathcal{F}(\vartheta)\tilde{\varphi}} \tag{36}$$

$$\Lambda_v(\tilde{\varphi},\vartheta) = \mu\phi_0[1-\mathcal{H}_v(\tilde{\varphi})]e^{-\mathcal{F}_v(\vartheta)\tilde{\varphi}} \tag{37}$$

$$\Pi(\tilde{\varphi},\vartheta) = \left\{\frac{\begin{array}{r}\bar{\lambda}[(\bar{\omega}+\omega\vartheta)\mu\phi_0[\bar{\mathcal{H}}_v(\mathcal{F}_v(\vartheta)) + \mathcal{V}(\vartheta)\bar{\mathcal{C}}(\mathcal{F}(\vartheta))] + (\bar{\omega}+\omega\vartheta)\lambda\gamma r_1\phi_0\bar{\mathcal{C}}(\mathcal{F}(\vartheta))\\ + \vartheta\bar{\lambda}\gamma r_1\phi_0\bar{\mathcal{I}}(\mathcal{F}(\vartheta)) - \mu\phi_0 - \gamma r_1\phi_0] + \vartheta\bar{\lambda}\gamma r_1\phi_0\end{array}}{\vartheta - [(\bar{\omega}+\omega\vartheta)\lambda\bar{\mathcal{C}}(\mathcal{F}(\vartheta)) + \vartheta\bar{\lambda}\bar{\mathcal{I}}(\mathcal{F}(\vartheta))][\vartheta + (1-\vartheta)\bar{\mathcal{B}}(\mu)]}\right\} \times [1-\mathcal{I}(\tilde{\varphi})]e^{-\mathcal{F}(\vartheta)\tilde{\varphi}} \tag{38}$$

3.5. Theorem

The stationary dist. of the no. of clients in the orbit while the server is free,normal operative service, slow service and the prob. that the server is idle is described by $\rho < \bar{\mathcal{B}}(\mu)$ under the stability condition

$$\chi(\vartheta) = \vartheta \left\{ \frac{\begin{array}{r}(\bar{\omega}+\omega\vartheta)\phi_0[\bar{\mathcal{H}}_v(\mathcal{F}_v(\vartheta)) + \mathcal{V}(\vartheta)\bar{\mathcal{C}}(\mathcal{F}(\vartheta))] + (\bar{\omega}+\omega\vartheta)\frac{\lambda}{\mu}\gamma r_1\phi_0\bar{\mathcal{C}}(\mathcal{F}(\vartheta)) \\ + \vartheta\frac{\bar{\lambda}}{\mu}\gamma r_1\phi_0\bar{\mathcal{I}}(\mathcal{F}(\vartheta)) - \phi_0 - \frac{\gamma r_1}{\mu}\phi_0\end{array}}{\vartheta - [(\bar{\omega}+\omega\vartheta)\lambda\bar{\mathcal{C}}(\mathcal{F}(\vartheta)) + \vartheta\bar{\lambda}\bar{\mathcal{I}}(\mathcal{F}(\vartheta))][\vartheta + (1-\vartheta)\bar{\mathcal{B}}(\mu)]} \right\} \times [1-\bar{\mathcal{B}}(\mu)] \tag{39}$$

$$\Psi(\vartheta) = \left\{ \frac{\begin{array}{r}\lambda[(\bar{\omega}+\omega\vartheta)\mu\phi_0[\bar{\mathcal{H}}_v(\mathcal{F}_v(\vartheta)) + \mathcal{V}(\vartheta)\bar{\mathcal{C}}(\mathcal{F}(\vartheta))] + (\bar{\omega}+\omega\vartheta)\lambda\gamma r_1\phi_0\bar{\mathcal{C}}(\mathcal{F}(\vartheta)) \\ + \vartheta\bar{\lambda}\gamma r_1\phi_0\bar{\mathcal{I}}(\mathcal{F}(\vartheta)) - \mu\phi_0 - \gamma r_1\phi_0][\vartheta + (1-\vartheta)\bar{\mathcal{B}}(\mu)] + \mu\phi_0\mathcal{V}(\vartheta) + \lambda\gamma r_1\phi_0\end{array}}{\vartheta - [(\bar{\omega}+\omega\vartheta)\lambda\bar{\mathcal{C}}(\mathcal{F}(\vartheta)) + \vartheta\bar{\lambda}\bar{\mathcal{I}}(\mathcal{F}(\vartheta))][\vartheta + (1-\vartheta)\bar{\mathcal{B}}(\mu)]} \right\} \times \frac{[1-\bar{\mathcal{C}}(\mathcal{F}(\vartheta))]}{\mu(1-\vartheta)} \tag{40}$$

$$\Lambda_v(\vartheta) = \frac{\mu\phi_0\mathcal{V}(\vartheta)}{\gamma} \tag{41}$$

$$\Pi(\vartheta) = \left\{ \frac{\begin{array}{r}\bar{\lambda}[(\bar{\omega}+\omega\vartheta)\mu\phi_0[\bar{\mathcal{H}}_v(\mathcal{F}_v(\vartheta)) + \mathcal{V}(\vartheta)\bar{\mathcal{C}}(\mathcal{F}(\vartheta))] + (\bar{\omega}+\omega\vartheta)\lambda\gamma r_1\phi_0\bar{\mathcal{C}}(\mathcal{F}(\vartheta)) \\ + \vartheta\bar{\lambda}\gamma r_1\phi_0\bar{\mathcal{I}}(\mathcal{F}(\vartheta)) - \mu\phi_0 - \gamma r_1\phi_0] + \vartheta\bar{\lambda}\gamma r_1\phi_0\end{array}}{\vartheta - [(\bar{\omega}+\omega\vartheta)\lambda\bar{\mathcal{C}}(\mathcal{F}(\vartheta)) + \vartheta\bar{\lambda}\bar{\mathcal{I}}(\mathcal{F}(\vartheta))][\vartheta + (1-\vartheta)\bar{\mathcal{B}}(\mu)]} \right\} \times \frac{[1-\bar{\mathcal{I}}(\mathcal{F}(\vartheta))]}{\mu(1-\vartheta)} \tag{42}$$

Proof. Taking the equations. $(35)-(38)$ and integrating them with regard to $\tilde{\varphi}$ and obtain the partial PGF's $\chi(\vartheta) = \int_0^\infty \chi(\tilde{\varphi},\vartheta)d\tilde{\varphi}$, $\Psi(\vartheta) = \int_0^\infty \Psi(\tilde{\varphi},\vartheta)d\tilde{\varphi}$, $\Lambda_v(\vartheta) = \int_0^\infty \Lambda_v(\tilde{\varphi},\vartheta)d\tilde{\varphi}$, $\Pi(\vartheta) = \int_0^\infty \Pi(\tilde{\varphi},\vartheta)d\tilde{\varphi}$,.

We can find the prob. that the server is free by using the normalisation condition (χ_0) and (ϕ_0) by establishing functions as, when there is no consumer in the orbit $\vartheta = 1$ in $(3.39)-(3.42)$ and using the "L'Hospital rule" if it is required, we examine $\chi_0 + \phi_0 + \chi(1) + \Psi(1) + \Lambda_v(1) + \Pi(1) = 1$. ■

3.6. Theorem

The stability constraint $\rho < \bar{\mathcal{B}}(\mu)$ used to determine the PGF of the no. of clients in the system and the orbit size dist. at a stationary point in time is given by

$$H_s(\vartheta) = \frac{Ne_s(\vartheta)}{De_s(\vartheta)} \tag{43}$$

$$H_0(\vartheta) = \frac{Ne_0(\vartheta)}{De_s(\vartheta)} \tag{44}$$

Proof. The "PGF of the no.of consumer in the system $(H_s(\vartheta))$ and in the orbit $(H_0(\vartheta))$" is calculated by applying $H_s(\vartheta) = \chi_0 + \phi_0 + \chi(\vartheta) + \vartheta\{\Psi(\vartheta) + \Lambda_v(\vartheta)\} + \Pi(\vartheta)$. and $H_0(\vartheta) = \chi_0 + \phi_0 + \chi(\vartheta) + \{\Psi(\vartheta) + \Lambda_v(\vartheta)\} + \Pi(\vartheta)$. Insert the eqns. $(39)-(42)$ in the earlier results,then the eqns. (43) and (44) may be computed immediately. ■

4. Measures of system performance

This part calculates many appropriate system prob., system efficiency metrics, and signifies the mean busy period and cycle that occur while the system is in various phases.

4.1. System state probabilities

By putting $\vartheta \to 1$ in equations. $(39) - (42)$ and applying "L Hospital's rule" wherever possible. we obtain the following findings.
(i)Pr(The server being available for the duration of the retrial)

$$\chi(1) = \phi_0[1-\bar{\mathcal{B}}(\mu)]\left\{\frac{[[\frac{\mu}{\gamma}[1-\bar{\mathcal{H}}_v(\gamma)] - \mu\mathcal{C}_1[1-\bar{\mathcal{H}}_v(\gamma)]] - \frac{\lambda}{\mu}\gamma r_1[\omega+\mu\mathcal{C}_1] + \frac{\bar{\lambda}}{\mu}\gamma r_1[1-\mu\mathcal{I}_1]}{\bar{\mathcal{B}}(\mu)+\lambda\omega-\lambda\mu\mathcal{C}_1-\bar{\lambda}(1+\mu\mathcal{I}_1)}\right\} \tag{45}$$

(ii)Pr(The server is operative on usual service period)

$$\Psi(1) = \left\{\frac{\phi_0\lambda\mathcal{C}_1[[[\frac{\mu}{\gamma}[1-\bar{\mathcal{H}}_v(\gamma)] - \mu\mathcal{C}_1[1-\bar{\mathcal{H}}_v(\gamma)]]] - \lambda\gamma r_1[\omega+\mu\mathcal{C}_1 + \bar{\lambda}\gamma r_1[1-\mu\mathcal{I}_1] + \mu[\mu\mathcal{H}_1+\frac{\mu}{\gamma}[1-\bar{\mathcal{H}}_v(\gamma)]]]}{\bar{\mathcal{B}}(\mu)+\lambda\omega-\lambda\mu\mathcal{C}_1-\bar{\lambda}(1+\mu\mathcal{I}_1)}\right\} \tag{46}$$

(iii)Pr(The server is on WV)

$$\Lambda_v(1) = \frac{\phi_0\mu[1-\bar{\mathcal{H}}_v(\gamma)]}{\gamma} \tag{47}$$

(iv)Pr(The server is under repair time during usual active period)

$$\Pi = \Pi(1) = \left\{\frac{\phi_0\bar{\lambda}\mathcal{I}_1[[[\frac{\mu}{\gamma}[1-\bar{\mathcal{H}}_v(\gamma)] - \mu\mathcal{C}_1[1-\bar{\mathcal{H}}_v(\gamma)]]] - \lambda\gamma r_1[\omega+\mu\mathcal{C}_1] + \bar{\lambda}\gamma r_1[1-\mu\mathcal{I}_1] + \bar{\lambda}\gamma r_1]}{\bar{\mathcal{B}}(\mu)+\lambda\omega-\lambda\mu\mathcal{C}_1-\bar{\lambda}(1+\mu\mathcal{I}_1)}\right\} \tag{48}$$

4.2. Average system size and its orbit

In a steady state, the system,
(i) Differentiating the equation (44) and the predicted no. of clients in the orbit (L_q) is established with regard to ϑ and $\vartheta = 1$.

$$L_q = H_o^{'}(1) = \lim_{\vartheta\to 1}\frac{d}{d\vartheta}H_o(\vartheta) = \phi_0\left[\frac{Ne_q^{'''}(1)De_q^{''}(1) - De_q^{'''}(1)Ne_q^{''}(1)}{3(De_q^{''}(1))^2}\right] \tag{49}$$

(ii) The predicted no. of clients in the system (L_s) is determined by differentiating the eqn. (43) with regard to ϑ and giving $\vartheta = 1$ yields.

$$L_s = H_s^{'}(1) = \lim_{\vartheta\to 1}\frac{d}{d\vartheta}H_s(\vartheta) = \phi_0\left[\frac{Ne_s^{'''}(1)De_q^{''}(1) - De_q^{'''}(1)Ne_q^{''}(1)}{3(De_q^{''}(1))^2}\right] \tag{50}$$

(iii) The mean waiting time of consumers in the system and queue $[W_s$ and $W_q]$ are computed utilizing "Little's method" $W_s = \frac{L_s}{\mu}$ and $W_q = \frac{L_q}{\mu}$ respectively.

4.3. Mean busy period and the busy cycle

Let $A(T_b)$ and $A(T_c)$ be the predicted sizes of the busy period and cycle, respectively under steady state conditions. The outcomes are derived directly from the justification of a different renewal procedure [5] , which concludes in

$$\phi_0 = \frac{A(T_0)}{A(T_b) + A(T_0)}; A(T_b) = \frac{1}{\mu}\left(\frac{1}{\phi_0} - 1\right); A(T_c) = \frac{1}{\mu\phi_0} = A(T_0) + A(T_b). \tag{51}$$

where T_0 is the period of time spent in the system's null state. Because there is an exponential difference in time between the arrivals of two customers and $A(T_0) = (1/\mu)$ with the parameter μ.

5. Particular Cases

We examine a few real-world examples of our technique that are consistent with the existing research in this area.
Case (i): No feedback,No VI and No SF
If $\omega = 1$,$\lambda = 1$, and $\gamma = 0$. The model may be lowered to a $M/G/1$ RQ with WV and the findings match those of Arivudainambi et.al.[14]
Case (ii): No retrial, No feedback and No starting failure.
Let $r_2 = 0$,$\omega = 1$, $\lambda = 1$ and $\bar{\mathcal{B}}(\mu) \to 1$. our framework has been simplified to an "$M/G/1$ queue with WVs and VI". Our results agree with Zhang and Hou [15].

6. Numerical Analysis

This section will demonstrate the different settings for system performance measures by using MATLAB. We investigate exponentially distributed retrial, service, slower pace service, vacation and repair periods. Numerical measurements are selected at random in order to fulfil the stability criteria. Tables 1 to 3 provides assessed outcomes of the idle prob., χ_0, ϕ_0 the "mean queue size (L_q), mean waiting time in the queue (W_q)" in our QM.
Table 1 shows that the retrial rate (η) escalates,χ_0 escalates, but L_q, W_q decreases for the value of $\omega = 0.19$, $\mu = 0.9$, $\lambda = 0.8$, $\gamma = 3$, $\bar{\mathcal{H}}_v(\gamma) = 0.9$, $r_1 = 0.5$.
Table 2 demonstrates that the vacation rate (γ) mounts,ϕ_0 increases,L_q, W_q subsides for the value of $\omega = 0.19$, $\mu = 1.5$, $\lambda = 0.19$, $\bar{\mathcal{H}}_v(\gamma) = 0.9$, $r_1 = 0.9$.
Table 3 clearly displays that feedback rate (ω) mounts, χ_0, L_q, W_q diminshes for the value of $\mu = 0.9$, $\lambda = 0.10$, $\bar{\mathcal{H}}_v(\gamma) = 0.9$, $r_1 = 0.19$, $\gamma = 0.9$.

Table 1: *The impact of Retrial rate (η) on χ_0, L_q, W_q*

Retrial rate (η)	χ_0	L_q	W_q
2.0	2.0718	0.0883	0.0982
2.5	2.2132	0.0885	0.0984
3.0	2.3172	0.0821	0.0912
3.5	2.3969	0.0728	0.0809
4.0	2.4599	0.0621	0.0690
4.5	2.5110	0.0507	0.0564
5.0	2.5532	0.0391	0.0434

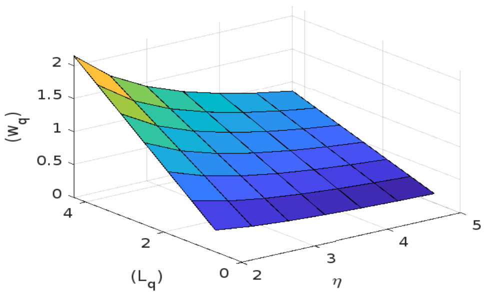
The Figure 1 (a) indicates that retrial rate (η) escalates, (L_q) and (W_q) increases. The Figure 1 (b) displays that vacation rate (γ) escalates, (L_q) and (W_q) decreases. The Figure 1 (c)

Table 2: *The impact of Vacation rate* (γ) *on* ϕ_0, L_q, W_q

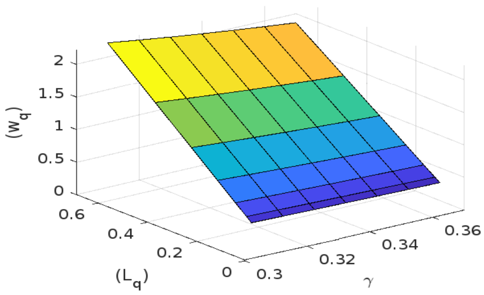
Vacation rate (γ)	ϕ_0	L_q	W_q
0.31	0.3724	0.6878	1.3756
0.32	0.3756	0.4931	0.9863
0.33	0.3785	0.3460	0.6921
0.34	0.3811	0.2890	0.4780
0.35	0.3835	0.1658	0.3316
0.36	0.3857	0.1211	0.2422
0.37	0.3878	0.1005	0.2010

Table 3: *The impact of Feedback rate* (ω) *on* χ_0, L_q, W_q

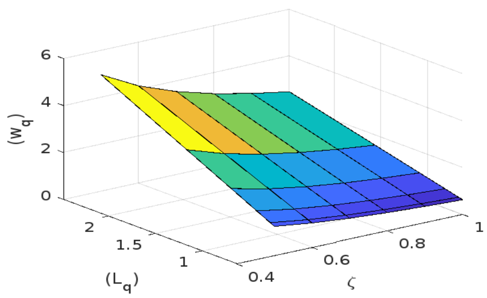
Feedback rate (ω)	χ_0	L_q	W_q
0.10	0.8306	0.2211	0.2457
0.20	0.8269	0.2008	0.2231
0.30	0.8233	0.1812	0.2013
0.40	0.8197	0.1623	0.1804
0.50	0.8161	0.1443	0.1603
0.60	0.8125	0.1270	0.1411
0.70	0.8090	0.1105	0.1227



(a) Retrial rate η Vs L_q, W_q



(b) Vacation rate γ Vs L_q, W_q.



(c) Feedback rate ω Vs L_q, W_q.

Figure 1: *Effects of a few parameters on 3D representation.*

demonstrates that feedback rate (ω) increases, (L_q) and (W_q) diminshes.

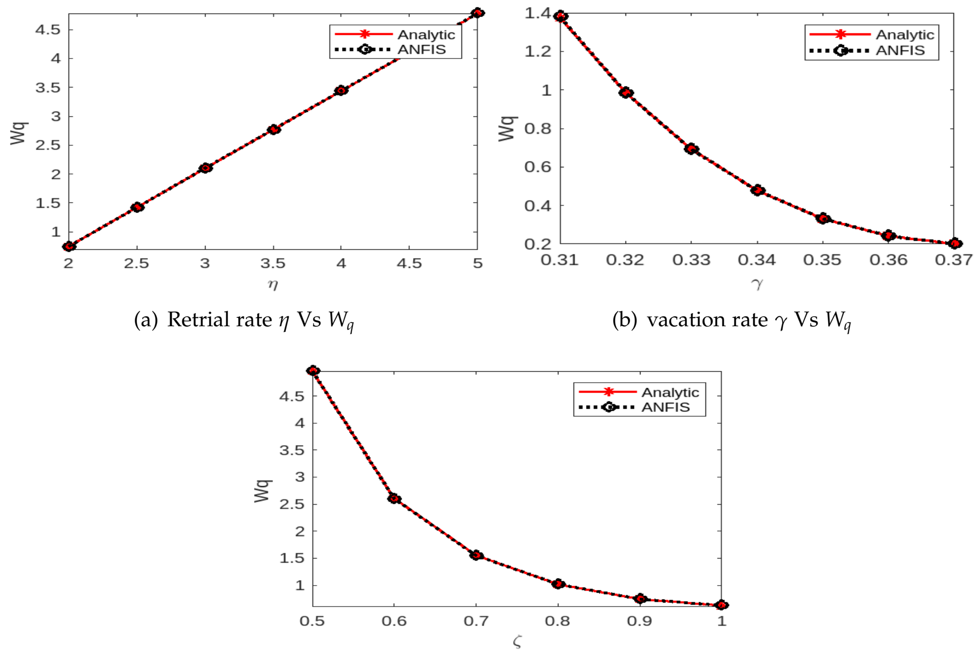
We may use the numerical findings above to determine the influence of features on the system's assessment criteria with certainty, the outcomes correspond to actual circumstances.

6.1. ANFIS Computing

Using the fuzzy toolbox of MATLAB software, an ANFIS network can be executed to compare outcomes from analyses. ANFIS, a soft computing approach, is an effective tool for identifying important results that are useful in busy everyday environments. This approach aids in the identification of approximate solutions for measurements whose definite outcomes would otherwise be difficult to determine.

In our framework, a neuro-fuzzy technique is used to compute the expected no. of consumers in the queue (W_q) by changing the retrial rate (η), vacation rate (γ) and feedback rate (ω)), as shown in the numerical results in Figure 2 $(a-c)$. We consider the parameters $(\eta, \gamma$ and $\omega)$ as linguistic variables (LV) that are performed for four epochs each. The analytic (ANFIS) outcomes are exhibited by the solid (dashed) lines. In the context of fuzzy systems, these factors are regarded as LV and are used as input variables in ANFIS networks.

The Gaussian function provides the membership functions for each of these input variables. The following are the linguistic values for each parameter: low, average, high, and excessive. The diagrams demonstrate agreement between the analytical findings for the paradigm and the neuro fuzzy results achieved through the ANFIS approach.



(a) Retrial rate η Vs W_q

(b) vacation rate γ Vs W_q

(c) Feedback rate ω Vs W_q

Figure 2: *Effects of a few parameters on 2D representation.(ANFIS)*

7. Cost Optimization

Our research aims to maintain system accessibility while optimizing system costs. Consequently, we establish the predicted cost function for system performance metrics and then accomplish a numerical analysis of the machining system under study. In order to calculate the best average cost per unit of time (TC), the parameters must be determined. This section discusses the best cost construct for the suggested approach using the standard cost notation form, and it provides the estimated total cost per unit of time as follows:

$$ETC = S_h L_s + S_b \Psi + S_v \Lambda + S_r \Pi + S_1 \zeta + S_2 \kappa \tag{52}$$

where,

- S_h = Holding cost per unit consumer.
- S_b = Cost per unit time while the server provides service during a usual busy period.
- S_v = Cost per unit time in the system when the server is on vacation.
- S_r = Cost per unit time for providing repair to the failed server.
- S_1 = Cost per unit time consumer served by the mean service rate ζ.
- S_2 = Cost per unit time consumer served by the mean vacation rate κ.

Equation (52) has an estimated total cost function that is multivariate and nonlinear. Therefore, developing an analytical solution for optimal parameter values say ζ^* and κ^* is problematic. In order to determine the most suitable numerical value for the decision parameters, the well-recognised meta heuristic technique: The optimization approach used is called Particle Swarm Optimization (PSO).

Choosing at random the present values for the cost element and the parameters are : $\mu = 0.05$, $\bar{B}(\mu) = 4.5$, $\bar{\lambda} = 5.9$, $\bar{H_v}(\gamma) = 0.75$, $(\zeta^* = 0.3670, \kappa^* = 0.3900)$.
Our goal is to identify the best values that will allow us to minimise the cost function. The five sets of cost factors that we have chosen are listed below.

Table 4: *Cost Sets values for different cost aspects*

Cost sets	S_h	S_b	S_v	S_1	S_2	S_r
Set 1	\$15	\$75	\$20	\$19	\$12	\$10
Set 2	\$20	\$85	\$25	\$17	\$19	\$23
Set 3	\$25	\$95	\$30	\$15	\$15	\$29

Applying the PSO algorithm using MATLAB software to the previously specified cost factors.
In this research, we have 100 candidates, 500 iterations throughout, and a range of parameters between 0.006 and 0.65 for the lower and upper bounds.
Tables 5 demonstrates that the effects of μ, ω, γ on TEC^* using PSO.

Table 5: *The PSO approach is executed by changing μ, ω and γ to determine the minimal cost for different cost sets.*

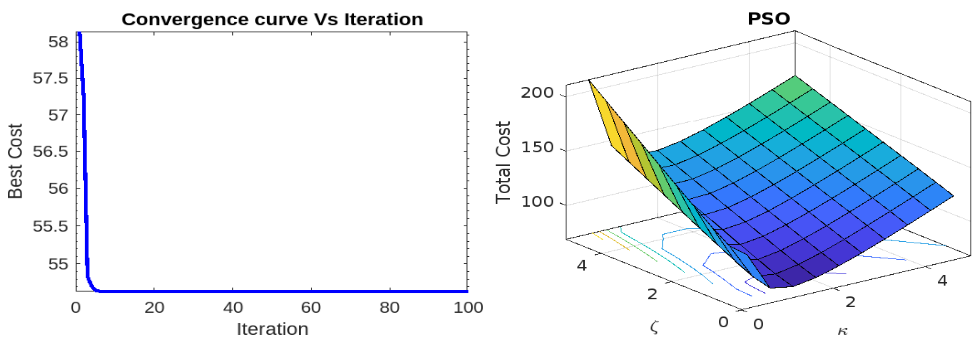
Parameters		(TEC^*)		
		Cost set 1	Cost set 2	Cost set 3
	0.20	\$75.1471	\$59.7276	\$66.7365
μ	0.25	\$115.1554	\$87.6434	\$92.1116
	0.30	\$151.7660	\$114.6657	\$119.3708
	1.00	\$73.5533	\$58.6272	\$65.7761
ω	1.15	\$74.5054	\$59.2872	\$66.3535
	1.20	\$74.8255	\$59.5074	\$66.5452
	1.25	\$78.6575	\$61.4278	\$67.9418
γ	1.30	\$76.0953	\$60.0313	\$66.8629
	1.35	\$73.5533	\$58.6272	\$65.7761

By changing a few of the variables, we were able to determine the entire system's cost, and we found that for $\mu = 1.00, \gamma = 1.35$ and $(\zeta^* = 0.3670, \kappa^* = 0.3900)$ the lowest cost was \$58.6272.

7.1. Particle Swarm Optimization

A precise evaluation of the QM is highly essential to offer adequate service and decrease congestion with the increasing expansion of computer networking and communications. If the greatest number of consumers can access an affordable system, then this is feasible. As a result, solutions that incorporate cost optimisation are very beneficial and advised. Jain et al. [25] and Jain and Meena [24] presented a cost investigation of a queueing model that includes an unreliable server and vacation periods. Utilizing the *particle swarm optimization* (PSO) approach, we have attempted to deal with the cost constraints in networking systems. This approach may sort through a very vast number of possible solutions to identify the most appropriate one. PSO has an additional benefit over a variety of optimisation strategies in that it does not require the objective function to be possible to differentiate.

Using this procedure, we first initiate a given population consisting of several particles or candidates. These candidates are then forced to travel inside the search space while adhering to the specified parameters and the goal function over their location and velocity. Every particle's fitness value is computed and to assess the values of the global best (gbest) and personal best (pbest) in more detail. The new gbest value is the particle whose pbest value is greater than gbest. This technique keeps on going until the predetermined number of iterations is reached. The algorithmic rule for PSO was first proposed by Kennedy and Eberhart [20]. The price optimization of a discrete-time RQ with SF utilizing this method has been examined by Upadhyaya [21]. Zhang et al.[22] examined set up cost and numerical answers for a single server recurrent model with state-dependent service using the PSO algorithmic approach. We have cited Malik et al.[23] investigation as it pertains to the operation of the PSO and GA algorithm.



(a) Convergence in PSO (b) Expected Total Cost

Figure 3: ***(a)****2D and 3D visualization of PSO optimization*

7.2. Convergence in PSO

Convergence holds significant importance within meta-heuristic optimization algorithms, representing the gradual improvement of potential solutions towards an optimal or near-optimal solution. The convergence pattern of an algorithm reflects its efficacy in exploring the solution space adeptly and moving closer to the global optimum. The findings from these figures suggest that employing the concept of a working vacation enhances the system's stability and reliability. This is due to the consistent availability of the server during this period. When a machine fails, the server responds quickly to the problem, but it provides a slower rate of service than when the machine is operating normally. Within PSO, particles converge toward the most optimal solution they are aware of, coming together as their movements become restricted and the finest solution steadies. Fig 3(a) Demonstrates that PSO achieves convergence towards the optimal cost. Fig 3(b)Displays the convexity and optimality of the cost function concerning the cost sets utilized in the optimization analysis.

8. Conclusion

In this research, a single server feedback retrial queueing system with starting failures, Bernoulli working vacations, and vacation interruptions was investigated. The number of customers in the system and its orbit are used to find the PGFs. This is done by using the "supplementary variable method". The average orbital queue length and the system average queue length have precise expressions. Numerical examples are used to verify the analytical conclusions. The mean busy period as well as other significant system performance indicators are determined. Also,numerical outcomes are compared to ANFIS. We have demonstrated how to optimize the functioning of a real-world service system using the PSO meta-heuristic algorithm. This suggested paradigm may be used in communication networks, supermarkets, management and production industries, etc. Basically, it is nearly impossible to construct a paradigm in which the server never defects or deactivates in any of these enormous sectors. As a result, this analysis is pertinent and in favour of scenarios in which a server can remain idle to maximise the consumption of resources. This model's construction helps to prevent the regular overcrowding issues that networking and communication systems suffer. The suggested model may be expanded in the future to incorporate other factors, such as modified vacation policy, randomized policy, consumer impatience , priorities, and setup times.

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