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ABOUT DEPOSIT DIVERSIFICATION PROBLEM

Possible ways to take into account risks, caused by uncertain factors, are investigated using the problem of optimal deposit diversification as an applied example. It is assumed that the investor (Decision Maker - DM) does not know future exchange rates at the end of the deposit period, and focuses only on some limits of their possible changes. Solution for this problem of decision-making under uncertainty depends on DM's attitude to the risk/income. Various solutions: optimal with respect to guaranteed income, optimal with respect to guaranteed risk (Savage minimax regret solution), as well as solution of multiple-criteria problem with two criteria of equal importance, namely, risk and income, are obtained.[1]

Keywords: minimax regret principle, uncertainty, maximin, risk, vector optimization.

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Introduction

Conditionally economists divide Decision Makers (DMs) into three categories according to their relation to risk and income (Cheremnykh 2008, Fisher, Dornbush and Schmalensee 1988). "Riskphobes"eliminate any risk and prefer to maximize the guaranteed income . "Riskphils"in their decisions take into account only the risks and seek to minimize them. The concept of risk is ambiguous. We should note that in this study the risk is understood as the risk by Savage - as a loss of income due to ignorance values of uncertain factors. "Neutral"DMs try to consider both indicators: the income and the risk. This leads to a problem of multiple-criteria decision making under uncertainty (MCDM). There are different approaches to its solution.

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1. Problem of formalization

In the beginning of some time period (to be specific - year) DM distributes \$1 to $m+1$ deposits in various currencies. Let $K_1..., K_m$ be courses of these currencies at the beginning of the year against the U.S. dollar (obviously the dollar's course $K_0 = 1$), and $x_0, x_1, \ldots, x_m$ be sizes of deposits in dollar terms. Interest rates of all types of the deposits $d_0 = r, d_1, \ldots, dm$ are supposed known. However, exchange rates at the end of the deposit period are unknown. This fact is reflected by uncertain parameters $y_1, ..., y_m$ (obviously, $y_0 = 1$). For these uncertain parameters only borders of their possible values are known:

$$y_1 \in [a_1, b_1], \ldots, y_m \in [a_m, b_m].$$

Financial result at the end of the year after conversion into dollars depends on both a plan of diversification $x = (x_0, x_1, \ldots, x_m)$ and the exchange rates at the end of the period — uncertainties $y_1, \ldots, y_m$:

$$f(x, y) = (1 + r)x_0 + \frac{(1 + d_1)x_1y_1}{K_1} + ... + \frac{(1 + d_m)x_my_m}{K_m}. \quad (1)$$

DM is interested in getting the greatest value of the final result $f(x, y)$. However, he should take into account the possibility of realizing any values of uncertain factors — the exchange rates $y_1 \in [a_1, b_1], ..., y_m \in [a_m, b_m]$.

Thus, the mathematical model of the problem of \$1 diversification is represented by triple

$$\Gamma =< X, Y, f(x, y) >,$$

where $X = \left\{x \in \mathbb{R}^{m+1} \middle| \sum_{i=0}^{m} x_i = 1, x_i \geq 0,\ (i = 0, \ldots, m)\right\}$ is a set of all admissible plans of diversification (a set of DM's strategies), $Y = [a_1, b_1] \times \cdots \times [a_m, b_m]$ is a set of possible values of uncertain vector $y = (y_1, \ldots, y_m)$, and $f(x, y)$ is an utility function (1) of depositor (DM). The value of this function will be called outcome.

From the standpoint of operations research, Γ is a single-criterion problem of decision making under uncertainty. At a fixed uncertainty y we are facing a problem of maximizing a linear function of x on a polyhedron X. The set X is a canonical simplex in $\mathbb{R}^{m+1}$.

Presence of uncertainty and desire to consider it leads to a concept of risk as a possibility of deviation of some results from their desired or expected values. DM's attitude to risk, willingness or unwillingness to consider it determines the type of decision-maker (Riskphobes , Riskphils, Neutral).

Decision-making in the problem Γ from the standpoint of all three grades constitutes the content of this paper.

2. The best guaranteed result (case of riskphobes)

Here we discuss a guaranteed solution for the DM, who does not accept (ignores) the risk, focuses only on outcomes and uses the principle of the best guaranteed result by Wald (maximin principle) (Wald 1939).

Definition 1. A pair (x^g, f^g) is referred as guaranteed on outcomes solution of the problem Γ if

$$f^g = \min_{y\in Y} f(x^g, y) = \max_{x\in X}\min_{y\in Y} f(x, y).$$

Maximin strategy x^g is a guarantying one and f^g is a guaranteed income. Construction of this solution consists of two stages:

Stage 1. Calculation of inner minimum (for every strategy $x \in X$) yields a guarantee

$$f[x] = \min_{y\in Y} f(x, y) = f(x, y(x)) \le f(x, y) \quad \forall y \in Y. \tag{2}$$

Stage 2. Calculation of outer maximum yields

$$f^g = f[x^g] = \max_{x\in X} f[x] \tag{3}$$

This stage yields the best (the biggest) guarantee because $f[x^g] \ge f[x]\ \forall x \in X$. The result of the stage 1 is

$$f[x] = f(x, a_1, \dots, a_m) = (1 + r)x_0 + \frac{(1 + d_1)x_1a_1}{K_1} + \dots + \frac{(1 + d_m)x_ma_m}{K_m} =$$

$$= \sum_{i=1}^{m}(1 + d_i)x_i\frac{a_i}{K_i} + (1 + r)x_0.$$

It follows from linearity $f(x, y)$ on variables $y_1, ..., y_m$ and special form of the set of uncertainties Y. The worst case of uncertainty $y(x) = a = (a_1, ..., a_m)$ is the same for every strategy $x \in X$, and the guarantee function $f[x]$ will be a linear function of $x_0, x_1, ..., x_m$ with coefficients $k_0 = (1 + r), k_1 = \frac{(1+d_1)a_1}{K_1}, ..., k_m = \frac{(1+d_m)a_m}{K_m}$. These coefficients define relative guaranteed effectiveness of various currencies deposits.

Hence the stage 2 is a special kind problem of Linear Programming (LP) with an objective function

$$f[x] = \sum_{i=0}^{m} k_ix_i$$

and with very special kind of linear constraints

$$\sum_{i=0}^{m} x_i = 1, \quad x_i \ge 0\ (i = 0, ..., m).$$

They define the set of all admissible strategies X as a polyhedron with $m+1$ vertices $x^{(1)} = (1, 0, 0.....0), x^{(2)} = (0, 1, 0, ..., 0), ..., x^{(m)} = (0, ..., 0, 1)$.

Remember one known in LP theory extreme property of a linear function on a polyhedron: maximal value is reached on some vertex of this polyhedron and a set of all points of maximum coincides with the convex linear hull of all vertices with maximal value of the function.

Therefore, $f^g = f[x^g] = \max_{x\in X} f[x] = \max_{0\le i\le m} f[x^{(i)}] = \max_{0\le i\le m} k_i$.

Let $R = \max_{0\le i\le m} k_i$ and I_R is a set of numbers of "the best"currencies with $k_i = R$. Then a set of all points of maximum for $f[x] = \sum_{i=0}^{m} k_i x_i$ on X will be

$$X^* = \left\{ x \in \mathbb{R}^{m+1} | \sum_{i=0}^{m} x_i = 1,\ x_i \ge 0\ (i \in I_R),\ x_i = 0\ (i \notin I_R) \right\}.$$

Proposition 1. The guaranteed on outcomes solution for the Problem Γ has the following form:

$$(x^g, f^g) = (x^g, R = \max_{0\le i\le m} k_i),$$

where

$$\begin{aligned} &x_i^g = 0,\ if\ k_i < R \\ &x_i^g \ge 0,\ if\ k_i = R = \max_{0\le i\le m} k_i,\ \sum_{i=0}^{m} x_i = 1. \end{aligned} \tag{4}$$

On the other words, DM should calculate coefficients $k_0 = (1 + r)$, $k_1 = = \frac{(1+d_1)a_1}{K_1}, ..., k_m = \frac{(1+d_m)a_m}{K_m}$, and deposit all \$1 to the currency with maximal value of coefficient $k_i = R$. If there are two or more such maximal coefficients, total sum may be distributed among the corresponding currencies in any arbitrary way.

3. Risk-oriented approach (Savage minimax regret solution)

This section relates to DM, who is oriented on minimization of guaranteed risk level. We shall use a principle of minimax regret by Savage (Savage 1954). To simplify calculations we consider later the problem with two currencies. Further notations are associated with previous ones as follows:

$x_0 = x,\ x \in [0, 1],\ x_1 = 1-x,\ y_1 = y, y \in [a, b],\ a_1 = a,\ b_1 = b,\ d_0 = r,\ d_1 = d, K_1 = K.$

Then the utility function has a form

$$f(x, y) = (1 + r)x + \frac{(1 + d)(1 - x)y}{K}. \tag{1'}$$

Definition 2. A pair (x^r, Φ^r) is referred as a guaranteed on risk solution of the problem Γ if $\Phi^r = \max_{y\in Y} \Phi(x^r, y) = \min_{x\in X} \max_{y\in Y} \Phi(x, y)$, where Savage risk function is

$$\Phi(x, y) = \max_{z\in X} f(z, y) - f(x, y).$$

Risk by Savage may be interpreted as a loss of the utility due to lack of knowledge of the uncertain parameters values at the moment of decision making.

Choosing the strategy $x \in X$, DM tries to minimize the guaranteed risk by Savage. Construction of this solution consists of three stages:

Stage 1: construction of functions $f[y] = \max\limits_{x \in X} f(x,y)$ and $\Phi(x,y) = f[y] - f(x,y)$.
Stage 2: computation of inner maximum – for every $x \in X$ the guarantee on risk $\max\limits_{y \in Y} \Phi(x,y) = \Phi[x] \geq \Phi(x,y)\ \forall y \in Y$ is defined.
Stage 3: computation of outer maximum – construction of $\Phi^r = \min_{x \in X} \Phi[x] = \Phi[x^r]$.

The second Stage for every strategy gives the guaranteed risk ($\Phi[x] \geq \Phi(x,y)\ \forall y \in Y$), on the third one the strategy with the least guaranteed risk is founded:

$$\Phi[x^r] = \Phi^r \leq \Phi[x]\ \forall x \in X \ and\ \Phi[x^r] \geq \Phi(x^r, y)\ \forall y \in Y.$$

Proposition 2. The guaranteed on risk solution of the problem Γ has the form

$$(x^r, \Phi^r) = \begin{cases} (1,0),\ if\ K\dfrac{1+r}{1+d} \geq b, \\ (0,0),\ if\ K\dfrac{1+r}{1+d} \leq a, \\ \left(\dfrac{K\frac{1+r}{1+d} - a}{b-a}, \dfrac{\left(b - K\frac{1+r}{1+d}\right)\left(K\frac{1+r}{1+d} - a\right)(1+d)}{K(b-a)} \right),\ if\ a < K\dfrac{1+r}{1+d} < b. \end{cases} \tag{5}$$

Proof. Let us consider 3 cases: first, $K\frac{1+r}{1+d} \geq b$, second, $K\frac{1+r}{1+d} \leq a$ and third $a < K\frac{1+r}{1+d} < b$. These cases overlay all possible variants of relative interposition of the point $K\frac{1+r}{1+d}$ and the segment $[a,b]$ on axis y.

Case 1. Let $K\frac{1+r}{1+d} \geq b \Rightarrow (1+r)K \geq b(1+d)$. Then

$$\begin{aligned} f(x,y) &= x(1+r) + (1-x)\frac{1}{K}(1+d)y \leq x(1+r) + (1-x)\frac{1}{K}(1+d)b \leq \\ &\leq x(1+r) + (1-x)\frac{1}{K}K(1+r) = 1+r \quad \forall x \in [0,1],\ \forall y \in [a,b]. \end{aligned}$$

But $f(1,y) = (1+r)$ for all $y \in [a,b]$. That is why with $K\frac{1+r}{1+d} \geq b$

$$\max_{x \in [0,1]} f(x,y) = f(1,y) = (1+r) = f[y]\ \forall y \in [a,b]\ (Stage\ 1).$$

In according with the Stages 2 and 3 $\Phi^r = \min\limits_{x \in [0,1]} \max\limits_{y \in [a,b]} \Phi(x,y) = 1 + r - \max\limits_{x \in [0,1]} \min\limits_{y \in [a,b]} f(x,y)$.

Due to the assumption $K\frac{1+r}{1+d} \geq b \geq a$ and the Proposition 1 we obtain $x^r = 1$ and $\min\limits_{y \in [a,b]} f(x^r, y) = 1 + r$. Hence, if $K\frac{1+r}{1+d} \geq b$, then

$$\Phi(x^r, y) = \Phi(1,y) = 1 + r - (1+r) = 0\ \forall y \in [a,b].$$

Therefore, $\Phi^r = \max\limits_{y \in [a,b]} \Phi(x^r, y) = 0$.

Case 2. Let $K\frac{1+r}{1+d} \leq a \Rightarrow (1+r)K \leq a(1+d)$. Just like the previous case for all $x \in [0,1],\ y \in [a,b]$ we have a chain of inequalities

$$f(x,y) = x(1+r) + (1-x)\frac{1}{K}(1+d)y \leq \frac{1+d}{K}x\left(\frac{1+r}{1+d}K - a\right) + \frac{1+d}{K}y \leq \frac{1+d}{K}y.$$

On the other hand, with $x = 0$ $f(0,y) = \frac{1+d}{K}y$. Then with $K\frac{1+r}{1+d} \leq a$ we have (Stage 1)

$$\max_{x\in[0,1]} f(x,y) = \frac{1+d}{K}y = f[y] = f(0,y)\ \forall y \in [a,b],$$

therefore

$$\Phi(x,y) = \frac{1+d}{K}y - x(1+r) - (1+d)(1-x)\frac{y}{K} = -\frac{1+d}{k}x\left[\frac{1+r}{1+d}K - y\right].$$

Later, with $K\frac{1+r}{1+d} \leq a$ we have

$$\Phi^r = \min_{x\in[0,1]}\max_{y\in[a,b]}\left\{-\frac{1+d}{K}x\left[\frac{1+r}{1+d}K - y\right]\right\} =$$

$$= \min_{x\in[0,1]}\left\{\begin{array}{r} 0, \quad x = 0 \\ -\sup\limits_{x\in(0,1]}\min\limits_{y\in[a,b]}\dfrac{1+d}{K}x\left[\dfrac{1+r}{1+d}K - y\right] \end{array}\right\} =$$

$$= \min_{x\in[0,1]}\left\{\begin{array}{r} 0, \quad x = 0 \\ -\sup\limits_{x\in(0,1]}\dfrac{1+d}{K}x\left[\dfrac{1+r}{1+d}K - b\right] = 0, \quad K\dfrac{1+r}{1+d} \leq a \end{array}\right\} = 0,$$

and $x^r = 0$

Case 3: $a < K\frac{1+r}{1+d} < b$. Let us calculate $f[y] = \max\limits_{x\in[0,1]} f(x,y)$ using inequalities $a(1+d) < (1+r)K,\ b(1+d) > (1+r)K$. As the function $f(x,y)$ linearly depends on x (with fixed y) its maximum is obtained in a boundary point of the segment $[0,1]$:

$$f[y] = \max_{x\in[0,1]} f(x,y) = \max\{f(0,y), f(1,y)\} = \max\{(1+r), \frac{1+d}{K}y\},$$

and $\Phi(x,y) = f[y] - f(x,y) = \max(\Phi_1, \Phi_2)$, where

$$\Phi_1(x,y) = (1+r) - (1+r)x - (1-x)\frac{1+d}{K}y = \frac{1-x}{K}[(1+r)K - (1+d)y],$$

$$\Phi_2(x,y) = \frac{1+d}{K}y - (1+r)x - (1-x)\frac{1+d}{K}y = \frac{x}{K}[(1+d)y - (1+r)K].$$

In according with Stage 2 and using the linearity of the functions $\Phi_1(x,y),\ \Phi_2(x,y)$ on y, we obtain $\forall x \in [0,1]$ the guarantee on risk

$$\Phi[x] = \max_{y\in[a,b]} \Phi(x,y) = \max_{y\in[a,b]} \max\left\{\Phi_1(x,y), \Phi_2(x,y)\right\} =$$

$$= \max\max_{y\in[a,b]}\left\{\Phi_1(x,y), \Phi_2(x,y)\right\} = \max\left\{\Phi_1(x,a), \Phi_2(x,b)\right\} = \max\left\{\Phi_1[x], \Phi_2[x]\right\} \tag{6}$$

where $\Phi_1[x]$, $\Phi_2[x]$ are linear functions:

$$\Phi_1[x] = \frac{1-x}{K}[(1+r)K - (1+d)a], \quad \Phi_2[x] = \frac{x}{K}[(1+d)b - (1+r)K]. \tag{7}$$

Hence, the function $\Phi[x] = \max\left\{\Phi_1[x], \Phi_2[x]\right\}$ is a piece-wise linear convex function with one break point. This point of interception x^r can be find from the condition $\Phi_1[x^r] = \Phi_2[x^r]$

$$x^r = \frac{1}{b-a|}\left[\frac{1+r}{1+d}K - a\right] \ (x^r \in (0,1) \textit{ due to Case } 3 \textit{ condition } a < K\frac{1+r}{1+d} < b).$$

As $\Phi_1[x]$ decreases and $\Phi_2[x]$ increases, x^r will be unique point of minimum for strong convex guaranteed risk function $\Phi[x]$ on $[0,1]$. Minimal guaranteed risk will be

$$\Phi^r = \min_{x\in[0,1]} \Phi[x] = \Phi[x^r] = \Phi_1[x^r] = \Phi_2[x^r] = \frac{\left(b - K\frac{1+r}{1+d}\right)\left(K\frac{1+r}{1+d} - a\right)(1+d)}{K(b-a)}. \tag{8}$$

Note that with Case 3 assumptions $\Phi^r > 0$.

4. Two-criterion approach (riskneytral case)

Suppose that DM takes into account both outcomes and risks, seeking to increase the value of the outcome $f(x,y)$ and reduce the value of the risk function $\Phi(x,y)$. At the same time DM should consider the possibility of any uncertainty $y \in Y$. Therefore we put into correspondence initial single-criterion problem Γ two-criterion problem under uncertainty:

$$\overline{\Gamma} = < X, Y, \{f(x,y), -\Phi(x,y)\}) >, \tag{9}$$

where $X, Y, f(x,y)$ are the same as in the problem Γ, and $\Phi(x,y)$ is the risk function.

In this problem DM, choosing his strategy $x \in X$, seeks to increase both criteria $f(x,y)$ and $-\Phi(x,y)$ (which corresponds in particular to reduction of risk $\Phi(x,y)$). Recall that the uncertainty y can take any value from the set $Y = [a,b]$.

Multiple-criteria problems under uncertainty were first studied in detail by Zhukovskiy and Molostvov in the paper (1980) and later in their monographs (1988, 1990). Various aspects of multiple-criteria optimization under uncertainty were investigated for both static and dynamic cases by Zhukovskiy and Salukvadze (1994), Salukvadze, Topchishvili and Zhukovskiy (2002), Molostvov (1983, 2004). The further development of the theory was carried out by Zhukovskiy and Kudryavtsev (2013). Following these results, introduce the concept of solution for the problem $\overline{\Gamma}$.

Definition 3. A triplet $(x^s, f^s, \Phi^s) \in X \times \mathbb{R}^2$ is referred as a strongly guaranteed on outcomes and risks solution (SGOR) of the problem $\overline{\Gamma}$, if:
a) there exists functions $y^{(i)} : [0,1] \to [a,b]$ $(i = 1,2)$ such that $\forall x \in [0,1]$

$$f^g[x] = \min_{y\in[a,b]} f(x,y) = f(x, y^{(1)}(x)), \ \Phi^g[x] = \max_{y\in[a,b]} \Phi(x,y) = \Phi(x, y^{(2)}(x)); \tag{10}$$

b) $\max\limits_{x\in[0,1]}(f^g[x]-\Phi^g[x]) = f^g[x^s]-\Phi^g[x^s]$;

c) $f^s = f^g[x^s],\ \Phi^s = \Phi^g[x^s]$. (11)

Remark 3. A game interpretation of the strongly guaranteed on outcomes and risks solution $x^s \in X$ is the following. Using the strategy $x^s \in X$, DM provides, with any possible realization of the uncertainty $y \in Y$, the guaranteed outcome $f^s \leq f(x^s, y)$ with the least possible risk $\Phi^s \geq \Phi(x^s, y)$ .If with any another strategy $x \in X$ the outcome will be better ($f(x, y) > f^s$), then in any case the risk will be worse ($\Phi(x, y) > \Phi^s$).

Calculation of the strongly guaranteed on outcomes and risk solution consists of three Stages.

Stage 1: construction of functions $f^g[x] = \min\limits_{y\in[a,b]} f(x, y), \Phi^g[x] = \max\limits_{y\in[a,b]} \Phi(x, y)$.

Stage 2: construction of Pareto-maximal strategy $x^s \in X$ in two-criterion «problem of guarantees» $\overline{\Gamma}_g =< X, f^g[x], -\Phi^g[x] >$ through maximization of linear convolution of two criteria with both coefficients equal 1.

Stage 3: construction with help of (11) guarantees on outcome f^s and risk Φ^s.

Lemma.

$$f^g[x^r] - \Phi^g_1[x^r] = \frac{2}{b-a}\left[\frac{1+r}{1+d}K - \frac{b+a}{2}\right]\left[\frac{1+r}{1+d}K - a\right] + \frac{1+d}{K}a,$$

where x^r and $\Phi_1[x^r] = \Phi^r$ were defined earlier in (5) and (8).

Proposition 3. The strongly guaranteed on outcomes and risks solution (SGOR) (x^s, f^s, Φ^s) for the Problem $\overline{\Gamma}$ has the following form:

$$(x^s, f^s, \Phi^s) = \begin{cases} (1, 1+r, 0),\ if\ K\frac{1+r}{1+d} \geq b, \\ \left(1, 1+r, \frac{1+d}{K}\left[b - \frac{1+r}{1+d}K\right]\right),\ if\ \frac{a+b}{2} < K\frac{1+r}{1+d} < b, \\ (x^r, f^r, \Phi^r),\ if\ a < K\frac{1+r}{1+d} \leq \frac{a+b}{2}, \\ \left(0, \frac{1+d}{K}a, 0\right),\ if\ K\frac{1+r}{1+d} \leq a, \end{cases}$$

where

$$x^r = \frac{1}{b-a}\left[K\frac{1+r}{1+d} - a\right],$$

$$f^r = f[x^r] = \frac{1+d}{K(b-a)}\left[K\frac{1+r}{1+d} - a\right]^2 + \frac{1+d}{K}a,$$

$$\Phi^r = \frac{\left(b - K\frac{1+r}{1+d}\right)\left(K\frac{1+r}{1+d} - a\right)(1+d)}{K(b-a)}.$$

Proof. Depending on relative interposition of the point $K\frac{1+r}{1+d}$ and the segment $[a, b]$ one should consider 4 cases.

Cases 1 and 4: $K\frac{1+r}{1+d} \geq b > a$ or $K\frac{1+r}{1+d} \leq a$. Here in appliance with the Propositions 1 and 2 will be:

$$\max_{x\in[0,1]}(f^g[x] - \Phi^g[x]) = \begin{cases} f^s = f^g[1] = 1+r, \ if \ K\dfrac{1+r}{1+d} \geq b > a, \\ f^s = f^g[0] = \dfrac{1+d}{K}a, \ if \ K\dfrac{1+r}{1+d} \leq a, \end{cases}$$

because $\Phi^s = \Phi^g[1] = \Phi^g[0] = 0$.

So SGOR in Cases 1 and 4 has the form:

$$(x^s, f^s, \Phi^s) = \begin{cases} (1, 1+r, 0), \ if \ K\dfrac{1+r}{1+d} \geq b, \\ \left(0, \dfrac{1+d}{K}a, 0\right), \ if \ K\dfrac{1+r}{1+d} \leq a. \end{cases}$$

Next two Cases are specified by the condition $a < K\frac{1+r}{1+d} < b$.

On the Stage 1 we will use the guarantee of the outcome

$$f^g[x] = \min_{y\in[a,b]} f(x,y) = x\frac{1+d}{K}\left[\frac{1+r}{1+d}K - a\right] + \frac{1+d}{K}a$$

and the guarantee of the risk

$$\Phi^g[x] = \max_{y\in[a,b]} \Phi(x,y) = \begin{cases} \Phi_1[x] = \dfrac{1-x}{K}[(1+r)K - (1+d)a], \ for \ x \in [0, x^r], \\ \Phi_2[x] = \dfrac{x}{K}[(1+d)b - (1+r)K], \ for \ x \in [x^r, 1]. \end{cases}$$

On the second Stage, to calculate the difference $f^g[x] - \Phi^g[x]$, formulas for $f^g[x]$ and $\Phi^g[x]$ from the Stage 1 are used. Namely, for $x \in [0, x^r]$

$$f^g[x] - \Phi_1^g[x] = 2x\frac{1+d}{K}\left[\frac{1+r}{1+d}K - a\right] + 2\frac{1+d}{K}a - 1 + r =$$
$$= 2x\frac{1+d}{K}\left[\frac{1+r}{1+d}K - a\right] - \frac{1+d}{K}\left[\frac{1+r}{1+d}K - 2a\right],$$

and for $x \in [x^r, 1]$

$$f^g[x] - \Phi_2^g[x] = 2x\frac{1+d}{K}\left[\frac{1+r}{1+d}K - \frac{a+b}{2}\right] + \frac{1+d}{K}a.$$

Then the problem Γ (where $X = [0,1]$) we associate a pair of two-criterion problems:

$$\Gamma_1 =< [0, x^r], f^g[x], -\Phi_1^g[x] >, \quad \Gamma_2 =< [x^r, 1], f^g[x], -\Phi_2^g[x] > .$$

In view of earlier obtained expressions for x^r and (8)

$$x^r = \frac{1}{b-a}\left[K\frac{1+r}{1+d} - a\right],$$

$$\Phi^r = \frac{\left(b - K\frac{1+r}{1+d}\right)\left(K\frac{1+r}{1+d} - a\right)(1+d)}{K(b-a)}.$$

Case 2: $a < K\frac{1+r}{1+d} \leq \frac{a+b}{2}$. A "maximizator"$x^s$ for the difference $f^g[x] - \Phi_1^g[x]$ for the Problem Γ_1 has the form:

$$x^s = argmax_{x\in[0,x^r]}\left\{2x\frac{1+d}{K}\left[\frac{1+r}{1+d}K - a\right] - \frac{1+d}{K}\left[\frac{1+r}{1+d}K - 2a\right]\right\} = x^r,$$

and by Lemma

$$f^g[x^r] - \Phi_1^g[x^r] = \frac{2}{b-a}\left[\frac{1+r}{1+d}K - \frac{b+a}{2}\right]\left[\frac{1+r}{1+d}K - a\right] + \frac{1+d}{K}a \geq 0.$$

Analogously the "maximizator"x^s for the difference $f^g[x] - \Phi_2^g[x]$ for the Problem Γ_2 will be $x^s = x^r$, and by Lemma

$$f^g[x^r] - \Phi_2^g[x^r] = \frac{2}{b-a}\left[\frac{1+r}{1+d}K - \frac{b+a}{2}\right]\left[\frac{1+r}{1+d}K - a\right] + \frac{1+d}{K}a \geq 0.$$

So, with $a < K\frac{1+r}{1+d} \leq \frac{a+b}{2}$ we have $x^s = x^r$ and

$$f^g[x^s] = f^s = \frac{1+d}{K(b-a)}\left[K\frac{1+r}{1+d} - a\right]^2 + \frac{1+d}{K}a,$$

$$\Phi^g[x^s] = \Phi^s = \frac{\left(b - K\frac{1+r}{1+d}\right)\left(K\frac{1+r}{1+d} - a\right)(1+d)}{K(b-a)}.$$

Finally, strongly guaranteed on outcomes and risks solution has the form:

$$(x^r, f^r, \Phi^r) = \left(\frac{1}{b-a}\left[K\frac{1+r}{1+d} - a\right], \frac{1+d}{K(b-a)}\left[K\frac{1+r}{1+d} - a\right]^2 + \frac{1+d}{K}a,\right.$$
$$\left.\frac{\left(b - K\frac{1+r}{1+d}\right)\left(K\frac{1+r}{1+d} - a\right)(1+d)}{K(b-a)}\right).$$

Case 3: $\frac{a+b}{2} < K\frac{1+r}{1+d} < b$.

As $\frac{a+b}{2} > a$, then for the Problem Γ_1 we have

$$x^s = arg\max_{x\in[0,x^r]}\left\{2x\left[\frac{1+r}{1+d}K - a\right]\right\} = x^r,$$

and by Lemma we get the following equality:

$$f^g[x^r] - \Phi_1^g[x^r] = 2\frac{1+d}{(b-a)K}\left[\frac{1+r}{1+d}K - a\right]^2 - \frac{1+d}{K}\left[\frac{1+r}{1+d}K - 2a\right].$$

Similarly for the Problem Γ_2:

$$x^s = arg\max_{x\in[x^r,1]}\left\{2x\left[\frac{1+r}{1+d}K - \frac{a+b}{2}\right]\right\} = 1 \quad (as\ \frac{a+b}{2} < K)$$

and

$$f^g[1] - \Phi_2^g[1] = 2\frac{1+d}{K}\left[\frac{1+r}{1+d}K - \frac{a+b}{2}\right] - \frac{1+d}{K}a.$$

Since $\max_{x\in[0,1]}(f^g[x]-\Phi^g[x])=\max(f^g[x^r]-\Phi_1^g[x^r],f^g[1]-\Phi_2^g[1])=$

$$=\max\Big\{2\frac{1+d}{(b-a)K}\left[\frac{1+r}{1+d}K-a\right]^2-\frac{1+d}{K}\left[\frac{1+r}{1+d}K-2a\right],$$

$$2\frac{1+d}{K}\left[\frac{1+r}{1+d}K-\frac{a+b}{2}\right]-\frac{1+d}{K}a\Big\},$$

the condition $\frac{a+b}{2}<K\frac{1+r}{1+d}<b$ implies the inequality $f^g[x^r]-\Phi_1^g[x^r]<f^g[1]-\Phi_2^g[1]$.

So in Case 3 ($\frac{a+b}{2}<K\frac{1+r}{1+d}<b$) we have $x^s{=}1$ and SGOP has the following form

$$(x^s,f^s,\Phi^s)=\left(1,1+r,\frac{1+d}{K}\left[b-\frac{1+r}{1+d}K\right]\right)$$

The proof of Proposition 3 is finished.

5. Conclusions

We investigated the problem of optimal structure of multi-currency deposit under uncertainty of future exchange rates. We had only the limits of possible changes for these uncertain parameters, any statistical characteristics are unavailable. We considered three possible approaches for accounting risk: complete elimination of any risk, minimization of expected losses due to the uncertainty, and multi-criteria approach, which takes into account both criteria - the outcome and the risk. These approaches correspond to the type of a decision-maker, namely to his/her attitude to the risk: adversary of risk, lover of risk or neutral DM. Propositions 1, 2 and 3 give an explicit form of the optimal solutions for the problem of the guaranteeing deposit diversification depending on values of the meaningful economic parameters r,d,K,a,b and on DM's attitude to the risk. After defining his/her type, the value $K\frac{1+r}{1+d}$ and the limits a and b of the uncertain parameter y, DM obtains, by applying the corresponding formula, a numerical value for his/her guaranteeing deposit strategy. By the way, Proposition 3 shows that use of a "risk"strategy (from Proposition 2) reduces both the guaranteed risk and at the same time increases the guaranteed outcome, thus "killing two birds with one stone."Two last cases have been restricted by two-dimensional considerations. General approaches, solution concepts are the same for the problem with n currencies, but explicit farm of the solution will be replaced by some algorithm associated with piece-wise linear programming technique. Other possible direction for further investigations concerns on a "mixed variant"of incompleteness of the information, when we know stochastic distributions of some nondetermined factors, but only to within uncertain parameters. Such problems can be considered by combining stochastic and maximin approaches.

References

[1] Cheremnykh, Y.N. *Microeconomics. Advanced level.* //Moscow State University, INFRA-M, Moscow (in Russian). 2008.

[2] Molostvov, V.S. *Multiple-criteria optimization under uncertainty: concepts of optimality and sufficient conditions. Theory and Practice of Multiple Criteria Decision Making.//* North-Holland: 91-105. 1983.

[3] Molostvov, V.S. *Savage's principle for non-cooperative games under uncertainty, Proceedings of the international scientific conference "Problems of control and power engineering".//* Tbilisi'. 38-39. 2004.

[4] Salukvadze, M.E., Topchishvili, A.L., and Zhukovskiy, V.I. *Solutions of differential games under uncertainty.//* Bulletin of Tbilisi International Center of Mathematics and Informatics (TICMI), 5: 20 - 22. 2002.

[5] Savage, L.Y. *The Foundation of Statistics.* //Willey, New York. 1954.

[6] Wald, A. *Contribution to the theory of statistical estimation and testing hypothesis.//* Annuals Math. Statist., 10: 299-326. 1939.

[7] Zhukovskiy, V.I., and Molostvov, V.S. *On Pareto-optimality in cooperative differential games with multi-valued objective functionals.//* Mathematical Methods in Systems Theory (collected papers of the Institute for System Studies, Moscow), 6: 96-108 (in Russian). 1980.

[8] Zhukovskiy, V.I., and Molostvov V.S. *Multiple-criteria Decision Making in Conditions of Uncertainty (Mnogokriterial'noe prinyatie resheniy v usloviyah neopredelennosti).//* International Research Institute of Management Sciences, Moscow (in Russian). 1988.

[9] Zhukovskiy, V.I., and Molostvov, V.S. V.S. (1990). *Multiple-criteria Optimization of Systems in Conditions of Incomplete Information (Mnogokriterial'naya optimizaciya sistem v usloviyah nepolnoy informacii).//* International Research Institute of Management Sciences, Moscow (in Russian). 1990.

[10] Zhukovskiy, V.I., and Salukvadze, M.E. *The Vector-valued Maximin.//* Academic Press, New York, NY. 1994.

[11] Zhukovskiy, V.I., and Kudryavtsev, K.N. *Balancing of conflicts under uncertainty, Part II, Analogue of maximin, Mathematical theory of games//* 5-2: 3-45 (in Russian). 2013.

О задаче диверсификации вклада

На примере задачи диверсификации вклада исследуются возможные пути учитывать риски, вызванные неопределенными факторами. Предполагается, что инвестор (лицо, принимающее решение — ЛПР) не знает обменный курс на конец срока вклада и ориентируется только на некоторые границы, внутри которых он может изменяться. Решение этой задачи зависит от того, как ЛПР относится к риску и прибыли. Возможные решения: оптимальное в смысле гарантированной прибыли, оптимальное в смысле гарантированного риска (минимаксное сожаление Сэвиджа), а также решение многокритериальной задачи с двумя равнозначными критериями, а именно величиной риска и прибыли.

Ключевые слова: принцип минимаксного сожаления, неопределенность, максимин, риск, векторная оптимизация.