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A study on fractional order theory in thermoelastic half-space under thermal loading

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In this study, the effect of fractional order derivative on a two-dimensional problem due to thermal shock with weak, normal and strong conductivity is established. The governing equations are taken in the context of Green and Naghdi of type III model (GNIII model) under fractional order derivative. Based on the Laplace and exponential Fourier transformations with eigenvalues approach, the analytical solutions has been obtained. For weak, normal and strong conductivity, the numerical computations for copper-like medium have been done and the results are shown numerically. The graphical results indicate that the effect of fractional order parameter has a major role on all physical quantities involved in the problem.

Keywords: eigenvalue approach, Green and Naghdi of type III model, fractional order generalized thermoelasticity

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Дробное исчисление и его применение для описания термоупругого полупространства в условиях тепловой нагрузки

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В работе показано влияние производной дробного порядка на решение двумерной задачи о тепловой нагрузке с низкой, нормальной и высокой проводимостью. Определяющие соотношения выбраны в рамках модели Грина–Нагди типа III (GNIII) для производной дробного порядка. На основе преобразования Лапласа и экспоненциального преобразования Фурье с использованием метода собственных значений получены аналитические решения. Выполнены численные расчеты для среды со свойствами меди в случае низкой, нормальной и высокой проводимости. Показано, что параметр дробного порядка оказывает существенное влияние на расчет физических величин в задаче.

Ключевые слова: метод собственных значений, модель Грина–Нагди типа III, теория обобщенной термоупругости с производной дробного порядка

1. Introduction

Many existing models of physical processes have been modified successfully by using the fractional calculus. A series of integral theories and fractional derivatives was created in the last half of the last century. Various approaches and definitions of fractional derivatives have become the main object of numerous studies [1, 2]. Recently, to investigate the anomalous diffusion, a considerable research effort has been expended, which is characterized by the frac-

tional time equation of wave diffusion by Kimmich [3] as in the form below

$$\rho c = k I^{\alpha} c_{,ii}, \quad i = 1, 2, 3, \quad (1)$$

where k is the diffusion conductivity, ρ is the mass density, c is the concentration and I^{α} is the fraction of Riemann–Liouville integral operator of order α . It introduced as a natural generalization of the well-known integral $I^{\alpha} f(t)$ repeated m times and wrote in the form of convolution type [4]

$$I^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^\alpha f(\xi) d\xi, & 0 < \alpha \leq 2, \\ f(t), & \alpha = 0, \end{cases} \quad (2)$$

where $\Gamma(\alpha)$ is the Gamma function. The fractional order of weak, normal and strong heat conductivity under generalized thermoelastic theory was applied by Youssef [5] in the following form

$$q_i + \tau_0 \frac{\partial q_i}{\partial t} = -KI^{\alpha-1} \nabla T, \quad 0 < \alpha < 2. \quad (3)$$

By using Taylor expansion of time-fractional order, Ezzat and El-Karamany [6] proposed a new fractional order generalized thermoelasticity model, which developed by Jumarie [7] as

$$q_i + \frac{\tau_0^\alpha}{\Gamma(\alpha+1)} \frac{\partial^\alpha q_i}{\partial t^\alpha} = -K \nabla T, \quad 0 < \alpha \leq 1. \quad (4)$$

The fractional order study of generalized thermoelastic problems is an important branch in solid mechanics [8–12]. In addition, Abbas [13] studied the effects of fractional order and magnetic field in a thermoelastic medium due to moving heat source using the eigenvalue approach. Sherief and Abd El-Latif [14] studied the effect of the fractional order parameter and the variable thermal conductivity on a thermoelastic half-space. Due to thermal source, the effect of fractional order parameter on plane deformation in a thermoelastic medium was studied by Kumar et al. [15]. Abbas and Youssef [16] studied a two-dimensional thermoelastic porous material under fractional order theory. The fractional order influence in a functional graded thermoelastic material problem has been solved by Abbas [17]. Youssef and Abbas [18] studied the theory of generalized thermoelasticity with fractional order derivative in the case of variable thermal conductivity. Based upon the theory of two-temperature generalized thermoelasticity, Zenkour and Abouelregal [19] investigated the fractional heat conduction for an unbounded medium with a spherical cavity. Abbas [20] studied the solution of thermoelastic diffusion problem under fractional order theory in an infinite elastic medium with a spherical cavity.

In this work, the eigenvalue approach has been used to obtain the analytical solutions for temperature, displacement and the stress components. By employing an analytical-numerical technique based on the eigenvalues approach with Laplace and Fourier transformations, the nondimensional equations have been handled. Numerical computations for copper-like medium have been done for strong, normal and weak conductivity and the effect of the fractional order parameter has been estimated.

2. Basic equations

Let us consider a homogeneous, thermoelastic isotropic half-space $y \geq 0$ at initial uniform temperature T_0 . Cartesian coordinate system (x, y, z) has been used with y axis is taken perpendicular to the bounding plane (Fig. 1). The

displacement vector has the form $\mathbf{u} = (u, v, 0)$. The governing equations have the following form

$$(\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} + (\lambda + \mu) \frac{\partial^2 v}{\partial x \partial y} + \mu \frac{\partial^2 u}{\partial y^2} - \gamma \frac{\partial T}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}, \quad (5)$$

$$(\lambda + 2\mu) \frac{\partial^2 v}{\partial y^2} + (\lambda + \mu) \frac{\partial^2 u}{\partial x \partial y} + \mu \frac{\partial^2 v}{\partial x^2} - \gamma \frac{\partial T}{\partial y} = \rho \frac{\partial^2 v}{\partial t^2}, \quad (6)$$

$$I^{\alpha-1} \left(K^* + K \frac{\partial}{\partial t} \right) \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) = \frac{\partial^2}{\partial t^2} \left(\rho c_e T + \gamma T_0 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right), \quad (7)$$

where the operator of fractional integral can be defined as the following [21]:

$$I^\alpha f(\varphi) = \frac{1}{\Gamma(\alpha)} \int_0^\varphi (\varphi - \varepsilon)^{\alpha-1} f(\varepsilon) d\varepsilon, \quad (8)$$

$$\begin{cases} 0 < \alpha < 1 & \text{for weak conductivity,} \\ \alpha = 1 & \text{for normal conductivity,} \\ 1 < \alpha \leq 2 & \text{for strong conductivity,} \end{cases}$$

$$\sigma_{xx} = (\lambda + 2\mu) \frac{\partial u}{\partial x} + \lambda \frac{\partial v}{\partial y} - \gamma(T - T_0), \quad (9)$$

$$\sigma_{yy} = (\lambda + 2\mu) \frac{\partial v}{\partial y} + \lambda \frac{\partial u}{\partial x} - \gamma(T - T_0), \quad (10)$$

$$\sigma_{xy} = \mu \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right), \quad (11)$$

where λ and μ are the elastic parameters, T is the increment of temperature, ρ is the density of mass, σ_{xx} , σ_{xy} and σ_{yy} are the stress components, T_0 is the body reference temperature, c_e is the specific heat at constant strain, K is the thermal conductivity, $\gamma = (2\lambda + 3\mu)\alpha_t$, and α_t is the linear thermal expansion coefficient. For convenience, the non-dimensional variables can be introduced on the following form:

$$\begin{aligned} (u', v, x', y') &= \frac{c}{\varsigma} (u, v, x, y), \quad T' = \frac{T - T_0}{T_0}, \\ (\sigma'_{xx}, \sigma'_{yy}, \sigma'_{xy}) &= \frac{1}{\mu} (\sigma_{xx}, \sigma_{yy}, \sigma_{xy}), \quad t = \frac{c^2}{\varsigma} t, \end{aligned} \quad (12)$$

where $c^2 = (\lambda + 2\mu)/\rho$, $\varsigma = K/(\rho c_e)$.

Fig. 1. Geometry of the problem

In terms of these nondimensional variables (12), Eqs. (5)–(11), after suppressing the primes, can be written as

$$\beta \frac{\partial^2 u}{\partial x^2} + (\beta - 1) \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial x^2} - \varpi \frac{\partial T}{\partial x} = \beta \frac{\partial^2 u}{\partial t^2}, \quad (13)$$

$$\beta \frac{\partial^2 v}{\partial y^2} + (\beta - 1) \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2} - \varpi \frac{\partial T}{\partial y} = \beta \frac{\partial^2 v}{\partial t^2}, \quad (14)$$

$$I^{\alpha-1} \left(\varepsilon_1 + \frac{\partial}{\partial t} \right) \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) = \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) \left(T + \varepsilon_2 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right), \quad (15)$$

$$\sigma_{xx} = \beta \frac{\partial u}{\partial x} + (\beta - 2) \frac{\partial v}{\partial y} + \frac{\partial^2 u}{\partial x^2} - \varpi T, \quad (16)$$

$$\sigma_{yy} = \beta \frac{\partial v}{\partial y} + (\beta - 2) \frac{\partial u}{\partial x} - \varpi T, \quad (17)$$

$$\sigma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad (18)$$

where $\beta = (\lambda + 2\mu)/\mu$, $\varpi = \gamma T_0/\mu$, $\varepsilon_1 = K^*/(\rho c^2 c_e)$, and $\varepsilon_2 = \gamma/(\rho c_e)$.

3. Application

Now, we can assume the homogeneous initial conditions as

$$\begin{aligned} T(x, y, 0) &= \frac{\partial T(x, y, 0)}{\partial t} = 0, \\ u(x, y, 0) &= \frac{\partial u(x, y, 0)}{\partial t} = 0, \\ v(x, y, 0) &= \frac{\partial v(x, y, 0)}{\partial t} = 0. \end{aligned} \quad (19)$$

The boundary conditions at $y = 0$ for the present problem are supposed as

$$T(x, 0, t) = T_1 H(\alpha - |x|) H(t), \quad (20)$$

where T_1 is a constant, H is the unit step function (Heaviside function). It means that the thermal shock acts on a band of a width 2α centered around the x axis on the half-space surface ($y = 0$) and is zero elsewhere. On the boundary $y = 0$, we suppose that the body displacement v does not depend on y , which leads to

$$\frac{\partial v(x, 0, t)}{\partial y} = 0. \quad (21)$$

Moreover, the medium is subjected to rigid foundation and rough enough to prevent the displacement u at any point of x and any time t , which leads to

$$u(x, 0, t) = 0. \quad (22)$$

4. Laplace and Fourier transformations

Let us define the Laplace transformation for a function $\Omega(x, y, t)$ by

$$L[\Omega(x, y, t)] = \bar{\Omega}(x, y, s) = \int_0^\infty \Omega(x, y, t) e^{-st} dt, \quad (23)$$

$s > 0$,

where s is a parameter. Hence, the above equations will take the form:

$$\beta \frac{\partial^2 \bar{u}}{\partial x^2} + (\beta - 1) \frac{\partial^2 \bar{v}}{\partial x \partial y} + \frac{\partial^2 \bar{u}}{\partial y^2} - \varpi \frac{\partial \bar{T}}{\partial x} = \beta s^2 \bar{u}, \quad (24)$$

$$\beta \frac{\partial^2 \bar{v}}{\partial y^2} + (\beta - 1) \frac{\partial^2 \bar{u}}{\partial x \partial y} + \frac{\partial^2 \bar{v}}{\partial x^2} - \varpi \frac{\partial \bar{T}}{\partial y} = \beta s^2 \bar{v}, \quad (25)$$

$$\frac{\partial^2 \bar{T}}{\partial x^2} + \frac{\partial^2 \bar{T}}{\partial y^2} = \frac{s^{\alpha-1}}{\varepsilon_1 + s} \left(\bar{T} + \varepsilon_2 \left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) \right), \quad (26)$$

$$\bar{\sigma}_{xx} = \beta \frac{\partial \bar{u}}{\partial x} + (\beta - 2) \frac{\partial \bar{v}}{\partial y} - \varpi \bar{T}, \quad (27)$$

$$\bar{\sigma}_{yy} = \beta \frac{\partial \bar{v}}{\partial y} + (\beta - 2) \frac{\partial \bar{u}}{\partial x} - \varpi \bar{T}, \quad (28)$$

$$\bar{\sigma}_{xy} = \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}, \quad (29)$$

$$\bar{T}(x, 0, s) = \frac{T_1 H(\alpha - |x|)}{s}, \quad (30)$$

$$\frac{\partial \bar{v}(x, 0, s)}{\partial y} = 0, \quad \bar{u}(x, 0, s) = 0.$$

Now, the exponential Fourier transformation for the function $\bar{\Omega}(x, y, s)$ can be given by

$$\bar{\Omega}^*(q, y, s) = \int_{-\infty}^\infty \bar{\Omega}(x, y, s) e^{-iqx} dx. \quad (31)$$

Hence, the above differential equations can assume the form:

$$-q^2 \beta \bar{u}^* + (\beta - 1) iq \frac{d\bar{v}^*}{dy} + \frac{d^2 \bar{u}^*}{dy^2} - iq \varpi \bar{T}^* = \beta s^2 \bar{u}^*, \quad (32)$$

$$\beta \frac{d^2 \bar{v}^*}{dy^2} + iq(\beta - 1) \frac{d\bar{u}^*}{dy} + q^2 \bar{v}^* - \varpi \frac{d\bar{T}^*}{dy} = \beta s^2 \bar{v}^*, \quad (33)$$

$$-q^2 \bar{T}^* + \frac{d^2 \bar{T}^*}{dy^2} = \frac{s^{\alpha+1}}{\varepsilon_1 + s} \left(\bar{T}^* + \varepsilon_2 \left(iq \bar{u}^* + \frac{d\bar{v}^*}{dy} \right) \right), \quad (34)$$

$$\bar{\sigma}_{xx}^* = iq \beta \bar{u}^* + (\beta - 2) \frac{d\bar{v}^*}{dy} - \varpi \bar{T}^*, \quad (35)$$

$$\bar{\sigma}_{yy}^* = \beta \frac{d\bar{v}^*}{dy} + iq(\beta - 2) \bar{u}^* - \varpi \bar{T}^*, \quad (36)$$

$$\bar{\sigma}_{xy}^* = \frac{d\bar{u}^*}{dy} + iq \bar{v}^*, \quad (37)$$

$$\bar{T}^*(q, 0, s) = T_1 \sqrt{\frac{2}{\pi}} \frac{\sin(q\alpha)}{qs}, \quad (38)$$

$$\frac{d\bar{v}^*(q, 0, s)}{dy} = 0, \quad \bar{u}^*(q, 0, s) = 0.$$

By using the eigenvalue approach as made in Refs. [22, 23], let us now proceed to solve the coupled differential equations (32)–(34). In the form of vector-matrix differential equation, Eqs. (32)–(34) can be written as follows

$$\frac{d\vartheta}{dy} = \mathbf{A}\vartheta, \quad (39)$$

where

$$\vartheta = \begin{bmatrix} \bar{v}^* & \bar{u}^* & \bar{T}^* & \frac{d\bar{v}^*}{dy} & \frac{d\bar{u}^*}{dy} & \frac{d\bar{T}^*}{dy} \end{bmatrix}^T$$

and

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ a_{41} & 0 & 0 & 0 & a_{45} & a_{46} \\ 0 & a_{52} & a_{53} & a_{54} & 0 & 0 \\ 0 & a_{62} & a_{63} & a_{64} & 0 & 0 \end{bmatrix}.$$

Thus, the matrix $\mathbf{A}$ has the characteristic equation in the form

$$\xi^6 - M_1\xi^4 + M_2\xi^2 + M_3 = 0, \quad (40)$$

where

$$\begin{aligned} M_1 &= a_{41} + a_{52} + a_{45}a_{54} + a_{63} + a_{46}, \\ M_2 &= a_{41}a_{52} - a_{53}a_{62} - a_{46}a_{54}a_{62} + a_{41}a_{63} + \\ &\quad + a_{52}a_{63} + a_{45}a_{54}a_{63} + a_{46}a_{52}a_{64} - a_{45}a_{53}a_{64}, \\ M_3 &= a_{41}a_{53}a_{62} - a_{41}a_{52}a_{63}. \end{aligned}$$

The eigenvalues of matrix $\mathbf{A}$ are the roots of Eq. (40) in the form $\pm\xi_1, \pm\xi_2, \pm\xi_3$. For the eigenvalue ξ , the corresponding eigenvector $\mathbf{Y} = [Y_1, Y_2, Y_3, Y_4, Y_5, Y_6]$ can be calculated as:

$$\begin{aligned} Y_1 &= -a_{45}a_{54}\xi + a_{46}\xi(a_{52} - \xi^2), \\ Y_2 &= -a_{46}a_{54}\xi^2 + a_{53}(a_{41} - \xi^2), \\ Y_3 &= -a_{41}a_{52} + (a_{41} + a_{52} + a_{45}a_{54})\xi^2 - \xi^4, \\ Y_4 &= \xi Y_1, \quad Y_5 = \xi Y_2, \quad Y_6 = \xi Y_3. \end{aligned} \quad (41)$$

For further reference, the following notations can be used:

$$\begin{aligned} Y_1 &= [\mathbf{Y}]_{\xi=-\xi_1}, \quad Y_2 = [\mathbf{Y}]_{\xi=-\xi_2}, \quad Y_3 = [\mathbf{Y}]_{\xi=-\xi_3}, \\ Y_4 &= [\mathbf{Y}]_{\xi=\xi_1}, \quad Y_5 = [\mathbf{Y}]_{\xi=\xi_2}, \quad Y_6 = [\mathbf{Y}]_{\xi=\xi_3}. \end{aligned} \quad (42)$$

The solution of Eq. (39) can be given in the following form:

$$\vartheta(q, y, s) = \sum_{i=1}^3 B_i Y_i e^{-\xi_i y}. \quad (43)$$

Due to the regularity conditions of the solution, the exponential in the space variable x has been discarded at infinity. By using the problem boundary conditions, the constants B_1, B_2 and B_3 have been calculated.

5. Double transformation inversion

The expression $\bar{\Omega}^*(q, y, s)$ in the domain of Fourier transformation can be expressed as

$$\bar{\bar{\Omega}}(q, y, s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{\Omega}^*(q, y, s) e^{iqx} dq. \quad (44)$$

Thus, the physical quantities can be given for x, y and s in the forms:

$$\bar{u}(x, y, s) = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^3 \int_{-\infty}^{\infty} B_k u_k e^{-\xi_k y + iqx} dq, \quad (45)$$

$$\bar{v}(x, y, s) = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^3 \int_{-\infty}^{\infty} B_k v_k e^{-\xi_k y + iqx} dq, \quad (46)$$

$$\bar{T}(x, y, s) = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^3 \int_{-\infty}^{\infty} B_k T_k e^{-\xi_k y + iqx} dq, \quad (47)$$

$$\begin{aligned} \bar{\sigma}_{xx}(x, y, s) &= \frac{1}{\sqrt{2\pi}} \sum_{k=1}^3 \int_{-\infty}^{\infty} B_k (iq\eta u_k - \\ &\quad - \xi_k(\eta - 2)v_k - \zeta T_k) e^{-\xi_k y + iqx} dq, \end{aligned} \quad (48)$$

$$\begin{aligned} \bar{\sigma}_{yy}(x, y, s) &= \frac{1}{\sqrt{2\pi}} \sum_{k=1}^3 \int_{-\infty}^{\infty} B_k (-\xi_k \eta v_k - \\ &\quad - iq(\eta - 2)u_k - \zeta T_k) e^{-\xi_k y + iqx} dq, \end{aligned} \quad (49)$$

$$\begin{aligned} \bar{\sigma}_{xy}(x, y, s) &= \frac{1}{\sqrt{2\pi}} \sum_{k=1}^3 \int_{-\infty}^{\infty} B_k (-\xi_k u_k + \\ &\quad + iqv_k) e^{-\xi_k y + iqx} dq, \end{aligned} \quad (50)$$

Fig. 2. Contour plots of the temperature distribution on the material for weak (a), normal (b) and strong conductivity (c) at $t = 0.12$

where u_k, v_k and T_k are the components of the corresponding eigenvectors. We adopt a numerical inversion method based on the Stehfest algorithm [24]. In this method, the inverse $g(x, y, t)$ of the Laplace transformation $\bar{g}(x, y, s)$ is approximated by the relation

$$\Omega(x, y, t) = \frac{\ln 2}{t} \sum_{j=1}^N V_j \bar{\Omega}\left(x, y, \frac{\ln 2}{t} j\right), \quad (51)$$

and V_j takes the form

$$V_j = (-1)^{(N/2+1)} \times \sum_{k=(i+1)/2}^{\min(i, N/2)} \frac{k^{(N/2+1)} (2k)!}{(N/2-k)! k! (i-k)! (2k-1)!}, \quad (52)$$

where N is the term number used in summation in Eq. (52) and should be optimized by trial and error.

6. Numerical results and discussion

In order to illustrate the theoretical results obtained in the previous section, we present some numerical values for the physical constants. We assume that the plate is made of an isotropic material (copper). The physical constants are listed below [25]:

$$\begin{aligned} \rho &= 8954 \text{ kg m}^{-3}, \lambda = 7.76 \times 10^{10} \text{ N m}^{-2}, \\ \mu &= 3.86 \times 10^{10} \text{ N m}^{-2}, \alpha_t = 1.78 \times 10^{-5} \text{ K}^{-1}, \\ T_0 &= 293 \text{ K}, c_e = 383.1 \text{ J kg}^{-1} \text{ K}^{-1}. \end{aligned}$$

Here the graphs are plotted to display the variations of temperature, displacements and stresses for $T_1 = 1$, $\alpha = 1$ at $t = 0.12$. The results of the numerical evaluation of temperature distribution contours for weak, normal and strong conductivity are illustrated in Fig. 2. We can observe that the change of temperature area is limited within a finite area and the temperature does not change outside that area. In addition, we can find that the heat influence zone with weak conductivity gets greater than the heat influence area with normal conductivity which is greater than the heat influence area with strong conductivity. Also, there are areas with a greater slope of the temperature of another zone. This means that heat conducts at a finite speed.

Figure 3 shows the variation of the nondimensional temperature, displacement components and stress components along the distance y for different fractional order parameter values $\alpha = 0.8, 1.0$ and 1.2 at fixed time ($t = 0.12$) when x direction ($x = 0.5$) remains constant. The solid

Fig. 3. The distribution of temperature T (a), horizontal u (b), vertical displacement v (c), σ_{xx} (d), σ_{xy} (e), and σ_{yy} (f) stress components with respect to y for different values of $\alpha = 0.8$ (1), 1.0 (2), 1.2 (3) when $x = 0.5$ at $t = 0.12$

Fig. 4 The distribution of temperature T (a), horizontal u (b), vertical displacement v (c) with respect to x for different values of $\alpha = 0.8$ (1), 1.0 (2), 1.2 (3) when $y = 0.5$ at $t = 0.12$

line (1) refers to the weak conductivity, while the dotted line (3) refers to the normal conductivity and the dashed line (2) refers to the strong conductivity. Figure 3, *a* represents the variation of temperature with the distance y . It is observed that at the boundary $y = 0$, the magnitude of temperature is equal to one, which agrees with the imposed boundary condition. The magnitude of temperature starts from the maximum value and then continuously decreases to zero with increasing y , while it increases with decreasing the fractional parameter α . Figure 3, *b* represents the variation of the horizontal displacement with the distance y . The horizontal displacement starts from zero which obey the boundary condition, then gradually increases until it attains a peak value at a particular location after that gradually decreases to zero with increasing y . The horizontal displacement increases with decreasing the fractional parameter. Figure 3, *c* shows the variation of the vertical displacement with the distance y . The magnitude of vertical displacement starts from the maximum value and then continuously decreases to close zero with increasing y . Also, before the intersection of the three curves, the vertical displacement magnitude decreases as the value of the fractional order decreases.

Figures 3, *d–f* depict stress components σ_{xx} , σ_{xy} and σ_{yy} with respect to the distance y . The magnitude of stress components σ_{xx} , σ_{xy} and σ_{yy} decreases as the value of the fractional order decreases before the intersection of the three curves. However, after the intersection, all values increase as the fractional order decreases.

Figure 4 demonstrate the variation of the nondimensional forms of temperature and the displacement components with the distance x for different fractional order parameter values $\alpha = 0.8, 1.0$ and 1.2 at fixed time ($t = 0.12$) when the y direction ($y = 0.5$) remains constant. The solid line (1) refers to the weak conductivity, while the dotted line (3) refers to the normal conductivity and the dashed line (2) refers to the strong conductivity.

In Fig. 4, *a* the temperature distribution is plotted along the distance x for $y = 0.5$. At the length of the surface heated ($-1 \leq x \leq 1$), the temperature field has maximum values and begins to decrease near the edges ($x = \pm 1$) where it decreases smoothly and finally closes zero. Figure 4, *b* displays the distribution of the horizontal displacement along the distance x . The vertical displacement starts decreasing at the beginning and ending of the surface heated ($-1 \leq x \leq 1$) and has a minimum value at the middle of the heated surface, after which it begins to increase and reaches a maximum close to the edges ($x = \pm 1$) and then decreases to zero. The variation of the vertical displacement with respect to x is shown in Fig. 4, *c*. The displacement magnitude has the maximum value along the length of the surface heated ($-1 \leq x \leq 1$), and it starts to decrease just near the edges ($x = \pm 1$), and then decreases to zero.

7. Conclusions

The effect of fractional order derivative on a two-dimensional problem due to thermal shock with weak, normal and strong conductivity has been investigated. The governing equations have been taken in the context of Green and Naghdi of type III model (GNIII model) under fractional order derivative. An analytical solutions has been obtained by employing the Laplace and exponential Fourier transformations in combination with the eigenvalues approach. For weak, normal and strong conductivity, some numerical computations for a copper-like medium have been carried out. The results demonstrate that the effect of fractional order parameter has a significant role for all physical quantities considered in the problem.

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